Problem 1.

The simplest way is to apply the tensor relations derived in class. Since there are only diagonal terms contributing in the system K, we get that the relations are very simple. For instance,

$$\begin{aligned} g^{'00} &= \left(g^{00} + \frac{v^2}{c^2} g^{11}\right)\gamma^2 \\ g^{'11} &= \left(g^{11} + \frac{v^2}{c^2} g^{00}\right)\gamma^2 \\ g^{'10} &= \left(-\frac{v}{c} g^{00} - \frac{v}{c} g^{11}\right)\gamma^2 \\ g^{'01} &= \left(-\frac{v}{c} g^{00} - \frac{v}{c} g^{11}\right)\gamma^2 \end{aligned} \tag{1}$$

and, using the fact that $g^{00} = -g^{11} = 1$, we get that these components of $g^{'\mu\nu}$ are equal to the same components of $g^{\mu\nu}$. The rest of the terms follow straightforwardly.

Another way of solving this problem is by considering Lorentz transformations as rotations in space-time and noticing that the tensor $g^{\mu}_{\nu} = \delta^{\mu}_{\nu}$. Therefore, it is invariant under Lorentz transformations.

An alternative way of solving this is noticing that

$$g^{'\mu\nu} = \frac{\partial x^{'\mu}}{\partial x^{\alpha}} \frac{\partial x^{'\nu}}{\partial x^{\beta}} g^{\alpha\beta} \tag{2}$$

and notice that the Lorentz transformations are such that

$$\frac{\partial x^{'\mu}}{\partial x^{\alpha}} = \frac{\partial x_{\alpha}}{\partial x'_{\mu}} \tag{3}$$

and, replacing above, and using $x^{\beta} = g^{\beta\alpha} x_{\alpha}$ we get

$$g^{'\mu\nu} = \frac{\partial x^{'\nu}}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x_{'\mu}} = \frac{\partial x^{'\nu}}{\partial x_{'\mu}} \tag{4}$$

But this is the derivative of the contravariant components with respect to the covariant components, that is just $g^{\mu\nu} = g^{\nu\mu}$

Problem 2.

a) The values of the electric and magnetic field outside the conductors may be obtained by applying the superposition law. The electric field is clearly zero, since there is no net charge density in the conductors. The magnetic field may be obtained by applying the inhomogeneous Maxwell equations and integrating over a section perpendicular to each conductor. For the magnetic field induced by each conductor one obtains

$$B_z^i 2\pi \Delta y_i = \frac{4\pi}{c} j_x a \tag{5}$$

$$B_z^i = \pm \frac{2v}{c} \frac{q}{L\Delta y_i} \tag{6}$$

where Δy_i is the difference between the y-coordinate of a point in the x-y plane and the position of each conductor (y and $y - D$, respectively).

The total magnetic field on a point in the plane determined by the conductors is just

$$B_z = -\frac{2qv}{cL} \left(\frac{D}{y(y-D)} \right) \tag{7}$$

The expression is only valid outside the conductors, and therefore the points $y = 0$ and $y = D$ are excluded.

The magnitude of the force acting upon a particle moving with velocity v in the x direction is simply $|B_z|v$ and certainly depends on y.

b) In the system K' the negative charges move at velocity $-v$. The positive charges of the first conductor are at rest, while the positive charges of the second conductor move with velocity

$$v' = -\frac{2v}{1 + v^2/c^2} \tag{8}$$

The distance between negative charges in both conductors in the system K' is equal to $L\sqrt{1 - v^2/c^2}$.

The distance between positive charges in the first conductor is equal to $L/\sqrt{1-v^2/c^2}$.

The distance between positive charges in the second conductor may be determined by considering that, with respect to the distance in the rest frame, in the system K the distance is contracted by a factor $\sqrt{1-v^2/c^2}$ while in the system K' it is contracted by a factor $\sqrt{1-(v')^2/c^2}$. Hence, the distance between positive charges in the second conductor is just equal to

$$L_2' = L\sqrt{\frac{1-(v')^2/c^2}{1-v^2/c^2}} \tag{9}$$

or, equivalently

$$L_2' = L\sqrt{\frac{(1-v^2/c^2)}{(1+v^2/c^2)^2}} \tag{10}$$

Then, the total charge density in the first conductor is equal to

$$\rho_1 = \frac{q}{La}\left(-\frac{1}{\sqrt{1-v^2/c^2}} + \sqrt{1-v^2/c^2}\right)$$

$$\rho_1 = -\frac{qv^2/c^2}{La\sqrt{1-v^2/c^2}} \tag{11}$$

The total charge density in the second conductor is, instead, equal to

$$\rho_2 = \frac{q}{La}\left(-\frac{1}{\sqrt{1-v^2/c^2}} + \frac{1+v^2/c^2}{\sqrt{1-v^2/c^2}}\right)$$

$$\rho_2 = -\rho_1 \tag{12}$$

c) The transformation Laws are equal to

$$\rho_1 = -\frac{v}{c^2}\frac{j_x}{\sqrt{1-v^2/c^2}}$$

$$\rho_1 = -\frac{v/c^2(\rho v)}{\sqrt{1-v^2/c^2}} \tag{13}$$

that agrees with the expression above. Similarly, ρ_2 is obtained by changing j_x by $-j_x$. Charge is conserved since $\rho_1 + \rho_2 = 0$

d) Again, the values of the electric and magnetic fields may be determined by applying the superposition law. The electric field at some point is

$$E'_y = \frac{2\rho_1 a}{\Delta y_1} + \frac{2\rho_2 a}{c\Delta y_2} \tag{14}$$

$$E'_y = \frac{2qv^2/c^2}{L\sqrt{1-v^2/c^2}} \frac{D}{y(y-D)} \tag{15}$$

In order to determine the magnetic field, we have to determine the currents in both conductors. Since there is no charge density in K, we obtain

$$j'_x = j_x/\sqrt{1-v^2/c^2} \tag{16}$$

in both conductors and therefore the magnetic field is simply

$$B'_z = B_z/\sqrt{1-v^2/c^2} \tag{17}$$

The relevant Lorentz transformations are

$$\begin{aligned} B'_z &= (B_z - v/cE_y)/\sqrt{1-v^2/c^2} \\ E'_y &= (E_y - v/cB_z)/\sqrt{1-v^2/c^2} \end{aligned} \tag{18}$$

and it is trivial to replace the above expressions to demonstrate that indeed, they are consistent with what we found above.

The force is simply equal to $F'y = qE'_y$ since the particle is at rest in the system K'.

Problem 3. The relevant transformation laws simply tell me that, since the density is zero and the current in the direction of y is equal to zero in K, then

$$\rho' = \gamma(\rho - v'j_y) = 0 \tag{19}$$

in each conductor. Similarly,

$$\vec{j'} = \vec{j} = j_x \hat{x} \tag{20}$$

The only complication in this problem is to consider the current transformations. First of all, the positive charge density in the system K' is affected by the fact that the transverse section of the conductor seems to be contracted by a factor $\sqrt{1-(v')^2/c^2}$ in that system. The distance L separating positive and negative charges is not affected, since it is perpendicular to the direction y. Hence, the positive and negative charge densities are just given by

$$\rho_+^{'} = \frac{q}{La\sqrt{1-(v')^2/c^2}} = -\rho_-^{'} \tag{21}$$

The total charge density is equal to zero, as anticipated.

The positive charge current in K' is equal to

$$j_x' = \rho_+^{'} u_x' = j_x \tag{22}$$

due to the fact that, since $u_y = 0$ in the system K, the velocity in the perpendicular x-direction is just contracted by a factor $\sqrt{1-(v')^2/c^2}$, while as shown above the density is enhanced by a factor $1/\sqrt{1-(v')^2/c^2}$. The negative charge current in the x-direction is still zero in K'.

By similar arguments, the currents of positive and negative charges in the system K' is simply given by

$$j_y' = \pm \frac{q}{La} \frac{v'}{\sqrt{1-(v')^2/c^2}} \tag{23}$$

respectively. Observe that the sum of the currents is still zero.