T Curtright & C Zachos, Rep Math Phys 76 (2015) 401 [1508.00868]

ELEMENTARY SU(3) GROUP ELEMENT !?!

▶ Group elements can be written as polynomials in Lie algebra generators.

Why bother? Consider $SU(2)$, the rotation matrix for 2-spinors,

$$\boxed{e^{ia(\hat{n}\cdot\vec{\sigma})} = I\cos a + i(\hat{n}\cdot\vec{\sigma})\sin a} \quad ,$$

so the generic SU(2) (SO(3)) group element multiplication is

$$e^{ia(\hat{n}\cdot\vec{\sigma})}e^{ib(\hat{m}\cdot\vec{\sigma})}$$

$$= I(\cos a\cos b - \hat{n}\cdot\hat{m}\sin a\sin b) + i(\hat{n}\sin a\cos b + \hat{m}\sin b\cos a - \hat{n}\times\hat{m}\ \sin a\sin b)\cdot\vec{\sigma}$$

$$= I\cos c + i(\hat{k}\cdot\vec{\sigma})\sin c = e^{ic(\hat{k}\cdot\vec{\sigma})}.$$

Manifestly, $\cos c = \cos a \cos b - \hat{n}\cdot\hat{m} \sin a \sin b$, the spherical law of cosines. Given c, $\rightsquigarrow$

$$\boxed{e^{ic\hat{k}\cdot\vec{\sigma}} = \exp\left(i\frac{c}{\sin c}(\hat{n}\sin a\cos b + \hat{m}\sin b\cos a - \hat{n}\times\hat{m}\ \sin a\sin b)\cdot\vec{\sigma}\right)}$$

✓**Composition law of finite rotations** —J W Gibbs, 1884.

⫛ A student asks me: "Is there something like that for $SU(3)$?"

I lied: "Aw... an awful mess...".

▶ **Not so**: The fundamental rep group element of $SU(3)$ is a quadratic in generators, quite compact and pretty.

Consider an arbitrary 3×3 traceless hermitian matrix H, so, e.g. an arbitrary linear combination of Gell-Mann matrices.

▶ The Cayley–Hamilton theorem gives

$$H^3 = I \det(H) + \tfrac{1}{2} H \operatorname{tr}\left(H^2\right) ,$$

and so $\det(H) = \operatorname{tr}\left(H^3\right)/3$.

$\looparrowright$ an H^2 term is absent in the polynomial expansion of H^3 because of $\operatorname{tr}(H) = 0$.

$\rightsquigarrow$ Since $\operatorname{tr}\left(H^2\right) > 0$ for any nonzero hermitian H, this bilinear trace factor may be absorbed into the normalization of H, thereby setting the scale of the group parameter space.

One may write the exponential of H as a matrix polynomial, quadratic in H by C–H, with polynomial coefficients that depend on the displacement from the group origin as a **rotation angle** θ.

The polynomial coefficients will also depend on invariants (traces, class functions) of the matrix H, which can be expressed in terms of the eigenvalues of H. For such a normalized H, there is effectively **only one invariant**: $\det(H)$.

This invariant may be encoded cyclometrically as yet **another angle**,

$$\phi = \frac{1}{3}\left(\arccos\left(\frac{3}{2}\sqrt{3}\det(H)\right) - \frac{\pi}{2}\right) \ ,$$

whose geometrical interpretation will clarify. Conversely,

$$\det(H) = -\frac{2}{3\sqrt{3}} \ \sin(3\phi) \ .$$

The quadratic invariant was absorbed into θ, so $\mathrm{tr}\left(H^2\right) = 2$, while the C–H thm expression collapses to $H^3 = H + I\det(H)$, all consistent with the Gell–Mann λ-matrices.

$\circledast$ H diagonalizes to $\frac{2}{\sqrt{3}}\mathrm{diag}((\sin\phi, \ \ \sin(\phi + 2\pi/3), \ \ \sin(\phi - 2p/3))$, by virtue of Viète's Z_3

$$x^3 - \frac{3}{4}x + \frac{1}{4}\sin 3\phi = (x - \sin\phi)\left(x - \sin(\phi + \frac{2\pi}{3})\right)\left(x - \sin(\phi - \frac{2\pi}{3})\right).$$

♦ ♦ For any $SU(3)$ group element generated by a traceless 3×3 hermitian matrix H ,

$$\exp(i\theta H) =$$

$$\sum_{k=0,1,2}\left[H^2+\frac{2}{\sqrt{3}}\ H\sin\left(\phi+\frac{2\pi k}{3}\right)-\frac{I}{3}\left(1+2\cos\left(2\left(\phi+\frac{2\pi k}{3}\right)\right)\right)\right]$$

$$\times\quad\frac{\exp\left(\frac{2}{\sqrt{3}}\ i\theta\sin\left(\phi+\frac{2\pi k}{3}\right)\right)}{1-2\cos\left(2\left(\phi+\frac{2\pi k}{3}\right)\right)}\ ,$$

where we have set the scale for the θ parameter space by choosing the normalization $\operatorname{tr}\left(H^2\right)=2$.

So expressed as a matrix polynomial, the group element depends on the sole invariant $\det(H)$ in addition to the group rotation angle θ. Both dependencies are in terms of elementary trigonometric functions when $\det(H)$ is expressed as the angle ϕ, whose geometrical interpretation follows immediately from the three eigenvalues of H exhibited above.

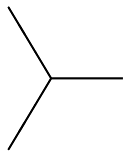
✓Check the **character**, the trace of the group element, $\exp\left(\frac{2}{\sqrt{3}}\ i\theta\sin(\phi)\right)+\exp\left(\frac{2}{\sqrt{3}}\ i\theta\sin\left(\phi-\frac{2\pi}{3}\right)\right)+\exp\left(\frac{2}{\sqrt{3}}\ i\theta\sin\left(\phi+\frac{2\pi}{3}\right)\right)$.

$\circlearrowleft$ These eigenvalues are the projections onto three mutually perpendicular axes of a single point on a circle formed by the intersection of the $0 = \mathrm{tr}\,(H)$ eigenvalue plane with the $2 = \mathrm{tr}\left(H^2\right)$ eigenvalue 2-sphere. The angle ϕ parameterizes that circle.

The eigenvalues are the projections onto a single axis of three points equally spaced on a circle (Viète): Z_3.

⊙ Two special cases

On the one hand, the **Rodrigues formula** for $SO(3)$ rotations about an axis $\hat{n}$, as generated by $j = 1$ spin matrices, is obtained for $\phi = 0 = \det(H)$,



$$\exp(i\theta H)|_{\phi=0} = I + iH\sin\theta + H^2(\cos\theta - 1) \ .$$

This is the Euler-Rodrigues result, upon identifying $H = \hat{n} \cdot \overrightarrow{J}$, which provides an explicit embedding $SO(3) \subset SU(3)$.

In fact, this holds if H is any *one* of the first seven Gell-Mann λ-matrices, or if H is a normalized linear combination of λ_{1-3}, or of λ_{4-7}.

(However, for generic linear combinations of λ_{1-7}, $\det(H)$ will *not* necessarily vanish.)

¶ On the other hand,

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

is the only one among Gell-Mann's choices for the 3×3 representation matrices for which $\phi \neq 0$, and for which two eigenvalues are degenerate. Clearly, $\det(\lambda_8) = \frac{-2}{3\sqrt{3}}$, so $\phi = \pi/6$. Moreover, $\lambda_8^2 = \frac{2}{3}\ I - \frac{1}{\sqrt{3}}\ \lambda_8$.

↬ Thus, directly from our formula,

$$\exp(i\theta\lambda_8) = \tfrac{1}{3}\left(2I + \sqrt{3}\lambda_8\right) e^{\frac{1}{3}i\sqrt{3}\theta} + \tfrac{1}{3}\left(I - \sqrt{3}\lambda_8\right) e^{-i\frac{2}{\sqrt{3}}\theta}$$

$$= \begin{pmatrix} \exp\left(i\theta/\sqrt{3}\right) & 0 & 0 \\ 0 & \exp\left(i\theta/\sqrt{3}\right) & 0 \\ 0 & 0 & \exp\left(-2i\theta/\sqrt{3}\right) \end{pmatrix},$$

as required .

♦ This particular example followed from the formula by carefully taking the limit as $\phi \to \pi/6$ of the $k = 0$ and $k = 1$ terms in that general expression, as necessitated by the degeneracy of the corresponding eigenvalues of λ_8), combined with the limit of the $k = 2$ term:

$$\lim_{\phi\to\pi/6}\left(\left[\lambda_8^2 + \tfrac{2}{\sqrt{3}}\ \lambda_8 \sin\left(\phi + \tfrac{2\pi}{3}\right) - \tfrac{1}{3}\ I\left(1 + 2\cos\left(2\left(\phi + \tfrac{2\pi}{3}\right)\right)\right)\right]\frac{\exp\left(\frac{2}{\sqrt{3}}\ i\theta \sin\left(\phi+\frac{2\pi}{3}\right)\right)}{1-2\cos\left(2\left(\phi+\frac{2\pi}{3}\right)\right)}\right.$$

$$= \lim_{\phi\to\pi/6}\left(\left[\lambda_8^2 + \tfrac{2}{\sqrt{3}}\ \lambda_8 \sin(\phi) - \tfrac{1}{3}\ I\,(1 + 2\cos(2\phi))\right]\frac{\exp\left(\frac{2}{\sqrt{3}}\ i\theta \sin\phi\right)}{1-2\cos(2\phi)}\right)$$

$$= \left(\tfrac{1}{3}\ I + \tfrac{1}{2\sqrt{3}}\ \lambda_8\right) e^{i\theta/\sqrt{3}}.$$

$\looparrowright$ One readily verifies that the Laplace transform of the formula furnishes the **resolvent** in the standard form as a matrix polynomial (Curtright, van Kortryk) involving the **adjugate** over the determinant,

$$\int_0^\infty e^{-t} \exp(itsH)\, dt = \frac{1}{I - isH} \ ,$$

$\looparrowright$

$$\frac{1}{I - isH} = \frac{1}{1 + s^2 + is^3 \det(H)} \left(\left(1 + s^2\right) I + isH - s^2 H^2 \right) \ .$$

$\leadsto$ The Laplace transform can be inverted to yield the formula in terms of the impulse response of the transfer function given by the prefactor in the resolvent,

$$\exp(i\theta H) = \left(H^2 - iH\frac{d}{d\theta} - I\left(1 + \frac{d^2}{d\theta^2}\right)\right) \sum_{k=0,1,2} \frac{\exp\left(\frac{2}{\sqrt{3}}\ i\theta \sin(\phi + 2\pi k/3)\right)}{1 - 2\cos 2(\phi + 2\pi k/3)} \ .$$

Also see T Curtright & T van Kortryk, *JPhys* **A48** (2015) 025202;

T Curtright, *J Math Phys* **56** 091703 (2015);

T Van Kortryk, "Matrix exponentials, SU(N) group elements, and real polynomial roots" [1508.05859]