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[arXiv:0806.3515] PhysLettB666 (2008) 386-390, T Curtright, D Fairlie & CZ; [arXiv:0903.4889] T Curtright, X Jin, L Mezincescu, D Fairlie, & CZ

BASICS OF TERNARY ALGEBRAS AND THEIR UNDERLYING NAMBU BRACKETS

Superposed M2-brane Lagrangean Model Chern-Simons interactions are predicated on Ternary algebras (CFZ, PhysLett B405 (1997) 37-44; Basu & Harvey, 2004; Bagger & Lambert, 2007)

Consider **associative** multiplication of 3 operators, fully antisymmetrized (Nambu 1973, Filippov 1984), the 3QNB,

$$\boxed{[A, B, C] \equiv A\,[B, C] + B\,[C, A] + C\,[A, B]\ .}$$

$= \frac{1}{2}\{A, [B, C]\} + \circlearrowleft$ $\rightsquigarrow$ The trinomial knows about **anticommutators**. Nontrivial trace.

It is related to, but, (**in sharp contrast** to 2N-QNBs vs. 2N-CNBs), it is **not a quantum deformation** of the also linear, antisymmetric 3CNB,

$$\{a, b, c\} = \epsilon^{ijk} \partial_i a \ \partial_j b \ \partial_k c = \frac{\partial(a, b, c)}{\partial(x, y, z)} \ ,$$

a Jacobian determinant (volume element).

▶ Will not consider the Awata-Li-Minic-Yoneya (1999) bracket,
$\langle A, B, C \rangle \equiv [A, B] \ \ \mathrm{Tr} C + [B, C] \ \ \mathrm{Tr} A + [C, A] \ \ \mathrm{Tr} B$,
repackaged commutators — traceless.

REVIEW OF A_4

$\circledcirc$ Nambu noted that the $su(2)$ Casimir appears in the 3QNB,

$$[L_x, L_y, L_z] \equiv L_x\,[L_y, L_z] + L_y\,[L_z, L_x] + L_z\,[L_x, L_y] = i\left(L_x^2 + L_y^2 + L_z^2\right) = iL^2\ .$$

$\rightsquigarrow$ the BL A_4 is in the enveloping algebra for $SU(2)$:

$$Q_x = \frac{L_x}{\sqrt[4]{L^2}}, \qquad Q_y = \frac{L_y}{\sqrt[4]{L^2}}, \qquad Q_z = \frac{L_z}{\sqrt[4]{L^2}}; \qquad Q_t \equiv \sqrt[4]{L^2},$$

yields

$$[Q_x, Q_y, Q_z] = iQ_t,\ [Q_t, Q_x, Q_y] = iQ_z,\ [Q_t, Q_y, Q_z] = iQ_x,\ [Q_t, Q_z, Q_x] = iQ_y.$$

Summarized as

$$[Q_a, Q_b, Q_c] = i\epsilon_{abc}{}^{d}\ Q_d\ ,$$

where $\epsilon_{xyzt} = +1$ with a $[-1, -1, -1, +1]$ Lorentz signature.

⋔ Amusing Aside: A_3

There is an even smaller 3QNB subalgebra, of this, namely

$$[Q_x, Q_y, Q_z \pm Q_t] = \pm i(Q_z \pm Q_t) \ ,$$

$\rightsquigarrow$ e.g.,

$$\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2+\sqrt{3} \end{pmatrix}\right] = \sqrt{3}\begin{pmatrix} 1 & 0 \\ 0 & 2+\sqrt{3} \end{pmatrix},$$

etc...

THE FI CONDITION

Filippov's (1984) special condition, "FI",

$$[D, E, [A, B, C]] = [[D, E, A], B, C] + [A, [D, E, B], C] + [A, B, [D, E, C]]$$

is not a general identity for associative operators, much unlike the Jacobi identity for commutators (2QNBs), and, unlike them, is **not** antisymmetric in all indices.

▶ But it **is an identity for 3CNBs**, and follows from the Leibniz rule of the **derivations** involved, and combinatorics: $\epsilon^{ab[c}\epsilon^{def]} = 0$
(TC & CZ, NewJPhys 4 (2002) 83.1-83.16).
It happens to be satisfied for ***very few*** 3QNB-based ternary algebras: for finite sets of operators, only A_4 and A_3; and for a few infinite ones (below). Is needed for BL model-building (consistency of supersymmetry). Defines Filippov n-Lie Algebras.

Informal preview:
⊛ **All such known ones are basically isomorphic to 3CNBs**

⟲ If you fully antisymmetrize 3QNB-into-3QNB, you get 5QNB, usually undefined (TC & CZ, PhysRev D68 (2003) 085001); and 3CNB-into-3CNB yields 5CNB.

↬ But if the FI is not "Jacobi", where is "Jacobi"?

BREMNER IDENTITIES

Bremner (1998, 2006), Nuyts (2008, unpublished) confirmed that there are no degree-5 (3QNB-into-3QNB) identitites; but found there is a degree-7 (3QNB-into-3QNB-into-3QNB) identitity,

$$\boxed{[[A,[B,C,D],E],F,G]=[[A,B,C],[D,E,F],G] \qquad \circlearrowleft} \ ,$$

$\circlearrowleft$: for fixed A, and antisymmetrized B,C,D,E,F,G.

A ternary algebra one might define through 3QNBs for associative chains of operators,

$$[T_a,T_b,T_c]=if_{abc}{}^{d}\;T_d \ ,$$

does not exist unless it satisfies this identity. (e.g. $\looparrowright$ constraining structure constants and centers).

$\rightsquigarrow$ If it does, it **need not satisfy the FI condition**—which is often harder to achieve.

AN EASY WAY TO SATISFY FI

In general checking FIs for a system is cumbersome—and the answer is usually negative...

▶ However, **if** a ternary algebra of 3QNBs shares structure constants with a ternary algebra of 3CNBs, **by identical combinatorics**, it will **also** satisfy the FI, because that is an identity for 3CNBs; which, in turn, likewise satisfy the Bremner identities, since the 3QNB version does.

⊛ One need not actually find an explicit associative operator realization of the 3CNB (analogous to the commutator realization of Poisson Brackets, $f(q,p) \mapsto \nabla f \times \nabla$ —very hard indeed!). One **only** need find an abstract ***shadow isomorph***: ***a 3CNB with the same structure constants***.

E.g., for finite-dimensional algebras, such as A_4,

$$Q_{x,y,z,t} \qquad \mapsto \qquad \sqrt{z}x, \quad \sqrt{z}y, \quad z, \quad \frac{x^2+y^2}{2} .$$

For A_3,

$$\sqrt{z}x, \qquad \sqrt{z}y, \qquad \sqrt{z}z.$$

Satisfy BI and FI. Comparably interesting ones in this class are some infinite-dimensional ones:

THE FOLLOWING SATISFY BI AND FI TOO

For the most general 3CNB on a T^3 basis, $e_a \equiv \exp(a \cdot (x, y, z))$, ..., up to normalization, the tetrahedron-volume algebra,

$$\boxed{[E_a, E_b, E_c] = a \cdot (b \times c)\, E_{a+b+c}}\ .$$

In fact, in the $\{e_a, e_b, e_c\}$ realization, it is **easier to check both** the FI and the BI!

Now taking a subalgebra, from 3-fold infinity of indices to a 2-fold one, e.g. for the **closing** set of functions,

$$w^a_m(x, y, z) \equiv \sqrt{z}\ \exp\left((a + 1/2)x + my\right),$$

yields the T^2 algebra of Chakraborty, Kumar, & Jain, also satisfying FI and BI.

▶ Finally, one understands the reason that the Virasoro-Witt ternary algebra with a 1-fold infinity, S^1, of indices, a smaller subalgebra of the tetrahedral volume one, satisfies the FI (BI evident).

This ternary algebra is put together by Witt (centerless Virasoro) and Goddard-Thorn operators,

$$[Q_k, Q_m, Q_n] = (k-m)\,(m-n)\,(k-n)\;\; R_{k+m+n}\ ,$$

$$[Q_p, Q_q, R_k] = (p-q)\;\; \left(Q_{k+p+q} + s\ k\ R_{k+p+q}\right)\ ,$$

$$[Q_p, R_q, R_k] = (k-q)\;\; R_{k+p+q}\ , \qquad\qquad [R_p, R_q, R_k] = 0\ ,$$

with s a parameter. Only for $s = \pm 2i$ is the FI satisfied.

We can understand this otherwise baffling fact as follows.
Simplify the algebra by $L_m \equiv Q_m + m\ \frac{s-\sqrt{s^2+4}}{2}\ R_m$, $\qquad \rightsquigarrow$

$$\boxed{[L_p, L_q, L_k] = 0\ , \quad [R_p, R_q, R_k] = 0\ , \quad [L_p, R_q, R_k] = (k-q)\;\; R_{k+p+q},}$$

$$\boxed{[R_p, L_q, L_k] = -\,(k-q)\;\; (L_{k+p+q} + p\sqrt{s^2+4}\ R_{k+p+q}).}$$

In fact, the square root may be absorbed in the relative normalizations of the Rs and Ls, and could be set to **one**; unless it were **zero**, which is thus a Wigner-Inonü contraction of that general case. This is the interesting case, satisfying the FI, and possessing a L–R $O(2)$ symmetry,

$$[L_p, L_q, L_k] = 0 \ , \qquad [R_p, R_q, R_k] = 0 \ ,$$

$$[L_p, R_q, R_k] = (k-q) \ R_{k+p+q} \ , \qquad [R_p, L_q, L_k] = -(k-q) \ L_{k+p+q} \ .$$

In this case, the closing set

$$\boxed{L_m = xe^{mz}, \qquad R_m = ye^{mz},}$$

identifies the underlying 3CNB algebra and so explains the FI.

▲ Are **all** FI-compliant ternary algebras really 3CNBs in disguise?