

Crib Notes on Campbell-Baker-Hausdorff expansions

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Consider/introduce auxiliary scaling parameter t , for counting powers, ultimately set to $t = 1$. For matrices, operators, etc, A and B ,

$$e^{tA}e^{tB} \equiv e^{Z(t,A,B)}.$$

Based on Lemmata 1,2, and 3, below,

$$Z \equiv \ln(e^{tA}e^{tB}) = (e^{tA}e^{tB} - 1) - \frac{1}{2}(e^{tA}e^{tB} - 1)^2 + \frac{1}{3}(e^{tA}e^{tB} - 1)^3 + \dots$$

is evaluated recursively [1] through algorithms in Applications 1,2,3,...

$$\begin{aligned} Z = t(A + B) + \frac{t^2}{2}[A, B] + \frac{t^3}{12} \left([[A, B], B] + [A, [A, B]] \right) + \frac{t^4}{24} [[B, A], A], B \\ - \frac{t^5}{720} ([[[[A, B], B], B], B] + [[[[B, A], A], A], A]) + \frac{t^5}{360} ([[[[A, B], B], B], A] + [[[[B, A], A], A], B]) \\ - \frac{t^5}{120} ([[[[A, B], B], A], B] + [[[[B, A], A], B], A]) + \dots \end{aligned}$$

- Powers of t higher than the first have coefficients which are *always commutators: Lie polynomials*—they are in the Lie Algebra. (Campbell, 1897; Poincaré, 1899. Structures Lie's converse (third fundamental) theorem: exponentiation of the algebra yields the simply connected group.) A concise proof of this was discovered by Eichler (1968). It follows that $\text{tr} \ln(e^A e^B) = \text{tr} A + \text{tr} B$.

- $Z(t, A, B) = -Z(-t, B, A)$, whence even powers of t are $A - B$ antisymmetric, while odd ones are symmetric.

- Thompson representation: $Z(t, A, B) = t \left(e^{-tW} A e^{tW} + e^{t\tilde{W}} B e^{-t\tilde{W}} \right)$, where W and \tilde{W} are in the Lie Algebra (A, B , and commutators), and $\tilde{W}(t, A, B) = W(-t, B, A) = \frac{A+B}{4} + \dots$.

- Zassenhaus expansion:

$$e^{t(A+B)} = e^{tA} e^{tB} e^{-\frac{t^2}{2}[A,B]} e^{\frac{t^3}{3!}(2[B,[A,B]]+[A,[A,B]])} e^{-\frac{t^4}{4!}([[[A,B],A],A]+3[[[A,B],A],B]+3[[[A,B],B],B])} \dots$$

Example: $e^{t(\partial+f'(x))} = e^{-f(x)} e^{t\partial} e^{f(x)} = e^{t\partial} e^{f(x)-f(x-t)} = e^{t\partial} e^{t f'(x)} e^{-\frac{t^2}{2!} f''(x)} e^{\frac{t^3}{3!} f'''(x)} e^{-\frac{t^4}{4!} f''''(x)} \dots$

- Triple formula:

$$e^{tV(t)} = e^{tA} e^{tB} e^{tA},$$

so that $V(t, A, B)$ is an even function of t , $V(t) = V(-t)$. Evaluated by, e.g., Application 1:

$$V = 2A + B + \frac{t^2}{6} [(A+B), [B,A]] + \dots$$

Note the duality between B and $-V$.

- Lie group commutator: $e^{tA}e^{tB}e^{-tA}e^{-tB} = e^U$.

$$U(t, A, B) = -U(t, B, A)$$

$$= t^2[A, B] + \frac{t^3}{2}[(A + B), [A, B]] + \frac{t^4}{3!}([[[B, A], A], B]/2 + [(A + B), [(A + B), [A, B]]]) \dots$$

Lemma 1: (Duhamel’s formula)

$$\delta e^Z = \int_0^1 ds e^{(1-s)Z} \delta Z e^{sZ},$$

for any matrix (noncommutative) Z . E.g., $\delta = \frac{\partial}{\partial t}$.

Provable directly from the finite N definition of the exponential, and then $N \rightarrow \infty$.

Alternative proof: Consider Δ as the operator δ acting on everything to its right.

$$\frac{d}{ds} \left(e^{-sZ} \Delta e^{sZ} \right) = e^{-sZ} [\Delta, Z] e^{sZ} ;$$

then integrate by $\int_0^1 ds$ to obtain

$$e^{-sZ} \Delta e^{sZ} - \Delta = \int_0^1 ds e^{-sZ} [\Delta, Z] e^{sZ},$$

hence

$$[\Delta, e^Z] = \delta e^Z = \int_0^1 ds e^{(1-s)Z} \delta Z e^{sZ}.$$

Lemma 2 (Hadamard formula): $e^A B e^{-A} = e^{[A, B]} \equiv B + [A, B] + \frac{1}{2!}[A, [A, B]] + \dots$

Proof: Note left commutation $[A, B] \equiv Ad(A) B$ acts on B like a derivative operator—obeys operator Leibniz’ chain rule. For a parameter s ,

$$\frac{d}{ds} \left(e^{sA} B e^{-sA} \right) = \left[A, e^{sA} B e^{-sA} \right],$$

so $f \equiv e^{sA} B e^{-sA}$ satisfies

$$\frac{df}{ds} = [A, f],$$

with B.C. $f(0) = B$. In turn, this is formally solved by the series in s , $f(s) = e^{s[A]} f(0)$.

Alternatively, it can be proved by induction in powers of s ,

$$\frac{d^n f}{ds^n} = [A, [A, [A, \dots, f] \dots]].$$

A direct consequence follows, $e^A e^B e^{-A} = e^{B + [A, B] + \frac{1}{2!}[A, [A, B]] + \dots}$, and hence the braiding relation,

$$e^A e^B = e^{B + [A, B] + \frac{1}{2!}[A, [A, B]] + \dots} e^A .$$

These two lemmata lead to

Lemma 3 (Campbell-Poincaré fundamental identity):

$$(\delta e^Z) e^{-Z} = \frac{e^{[Z]} - 1}{[Z]} \delta Z,$$

or equivalently,

$$\delta Z = \frac{[Z]}{e^{[Z]} - 1} \left((\delta e^Z) e^{-Z} \right),$$

where the fraction is the celebrated generating function of the Bernoulli numbers. Other equivalent forms are,

$$e^{-Z} \delta e^Z = \frac{1 - e^{-[Z]}}{[Z]} \delta Z = \delta Z \frac{e^{[Z]} - 1}{[Z]},$$

etc. So suitable choices for simple δe^Z s lead to determination of δZ and hence Z .

Application Algorithm 1. (Poincaré. Readily exhibits Z construction out of nested commutators, and applied mathematicians and Lie Group textbooks like it, cf. [2], but clumsy computationally).

Set now $e^{Z(t)} = e^A e^{tB}$. For $\delta Z = \partial_t Z \equiv Z'$ in Lemma 3, note that

$$B = e^{-Z} \delta e^Z = \frac{1 - e^{-[Z]}}{[Z]} Z',$$

hence

$$Z' = \frac{[Z]}{1 - e^{-[Z]}} B = \psi(e^{[Z]}) B,$$

where

$$\psi(x) \equiv \frac{x \ln x}{x - 1} = 1 - \sum_{n=1}^{\infty} \frac{(1-x)^n}{n(n+1)}.$$

(NB. $\psi(e^{-y}) = \sum_{n=0}^{\infty} B_n \frac{y^n}{n!}$, for the Bernoulli numbers $B_n(1)$: $B_0 = 1, B_1 = \frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, \dots$)

Since, from Lemma 2, $e^{[Z]} = e^{[A] e^{t[B]}}$, this reads

$$Z'(t) = \psi\left(e^{[A] e^{t[B]}}\right) B,$$

and so one may integrate over t from $Z(0) = A$, to finally obtain

$$Z(1) = A + \left(\int_0^1 dt \psi\left(e^{[A] e^{t[B]}}\right) \right) B.$$

It is manifest that all subleading terms to $A + B$ are commutators, so Z is in the Lie Algebra.

This is essentially an algorithm to produce the series by judicious power expansion of ψ and its arguments. For example, if only the term linear in B is sought (e.g.[3]), trivially, then,

$$Z = A + \psi(e^{[A]}) B + O(B^2) = A + \frac{[A]}{1 - e^{-[A]}} B + O(B^2),$$

where all expansion coefficients are simply related to the Bernoulli numbers as above.

Corollary: If $O(B^2)$ terms vanish, e.g. by virtue of special relations such as $[A, B] = sB$, it follows that $Z = A + \frac{s}{1-\exp(-s)}B = A + \psi(e^s)B$.

Taking inverses and rescaling yields a braiding relation, $e^A e^B = e^{(\exp s)B} e^A$. Thus, the group commutator amounts to just $e^A e^B e^{-A} e^{-B} = e^{(e^s-1)B}$.

Similarly, for $A = t\partial$, $B = f(x) - f(x-t) = (1 - \exp(-[A]))f(x)$, so that $[B, [A, B]] = 0 = [B, [A, [A, B]]] = \dots$, this exact expansion collapses to just $Z = A + [A, f(x)] = t(\partial + f'(x))$, the example employed for the above Zassenhaus expansion.

Application Algorithm 2. By virtue of its mechanical recursiveness, this one is favored by physicists, e.g. [4]. Set $e^{Z(t)} \equiv e^{tA} e^{tB}$. Operate both sides by $\delta = \partial_t$, and multiply by e^{-Z} on the right. By Lemmata 3, and 2,

$$\frac{e^{[Z]-1}}{[Z]} Z' = A + e^{t[A]} B.$$

Now, setting

$$Z \equiv \sum_{n=1}^{\infty} t^n Z_n,$$

the Z_n s can be solved for recursively in the power of t^{n-1} components of this eqn, so

$$Z_1 = A + B \implies Z_2 \implies Z_3 \implies \dots$$

Manifestly, again, for $n > 1$, each Z_n is a function of commutators only.

- $Z(t, A, B) = -Z(-t, B, A)$, \implies even powers of t are $A - B$ antisymmetric, while odd ones are symmetric.

Application Algorithm 3. (Hausdorff, cf [5]. Most systematic as a power expansion in A or B .)

Consider the replacement operators

$$\delta_A \equiv \left(\delta A \frac{\partial}{\partial A} \right), \quad \delta_B \equiv \left(\delta B \frac{\partial}{\partial B} \right),$$

which act on functions of A and B to successively replace each occurrence of A by δA , to first order, preserving the orderings, in accord with Leibniz's rule. Seek a symmetry of $Z(A, B)$, upon infinitesimal dilation of B , $\delta B = \epsilon B$, i.e. find $\delta A = -\epsilon D(A, B)$ s.t., to $O(\epsilon^2)$,

$$Z(A, B) = Z(A - \epsilon D, B + \epsilon B) + O(\epsilon^2),$$

so that

$$e^A e^B = e^{A-\epsilon D} e^{B+\epsilon B} = e^A (1 - \epsilon e^{-A} \delta_A e^A) (1 + \epsilon B) e^B + O(\epsilon^2).$$

So, evaluating $\delta_A e^A$ by Lemma 3, one has to $O(\epsilon^2)$,

$$\frac{1 - e^{-[A]}}{[A]} D = B,$$

whence

$$D = \frac{[A]}{1 - e^{-[A]}} B.$$

Consequently, $\delta Z = \delta_A Z + \delta_B Z = 0$ amounts to

$$\left(\left(\frac{[A]}{1 - e^{-[A]}} B \right) \frac{\partial}{\partial A} \right) Z = \left(B \frac{\partial}{\partial B} \right) Z.$$

The l.h.side raises the power of B , so the eqn may be solved recursively in each term Z_n of $O(B^n)$ in Z ,

$$Z_n = \frac{1}{n} \left(\left(\frac{[A]}{1 - e^{-[A]}} B \right) \frac{\partial}{\partial A} \right) Z_{n-1},$$

that is,

$$Z_0 = A, \quad Z_1 = \frac{[A]}{1 - e^{-[A]}} B = D, \quad \dots$$

etc, as in Application 1.

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