*Reprint of updated First Chapter of the Volume*

# QUANTUM MECHANICS IN PHASE SPACE

## An Overview with Selected Papers

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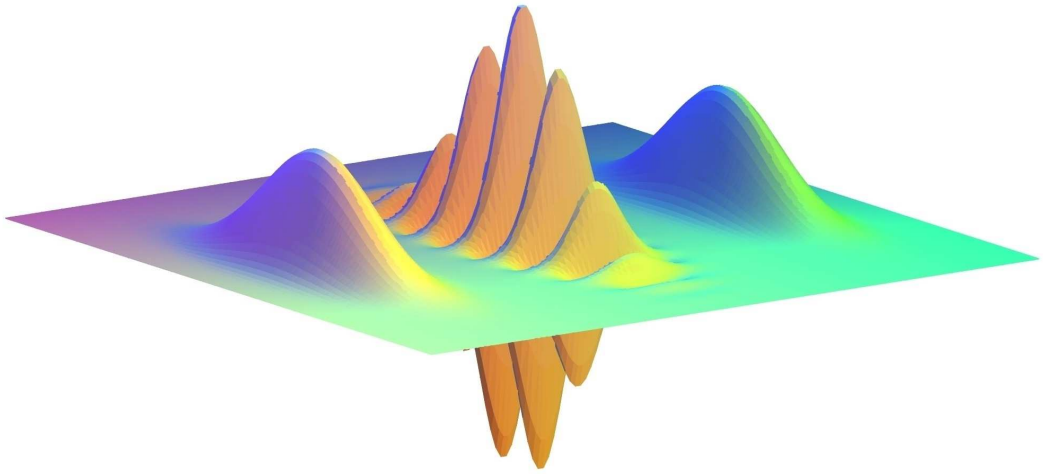
# Quantum Mechanics in Phase Space



Zachos, Fairlie, & Curtright

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## PREFACE

Wigner's quasi-probability distribution function in phase-space is a special (Weyl-Wigner) representation of the density matrix. It has been useful in describing transport in quantum optics; nuclear physics; and quantum computing, decoherence, and chaos. It is also of importance in signal processing, and the mathematics of algebraic deformation. A remarkable aspect of its internal logic, pioneered by Groenewold and Moyal, has only emerged in the last quarter-century: It furnishes a third, alternative, formulation of quantum mechanics, independent of the conventional Hilbert space, or path integral formulations.

In this logically complete and self-standing formulation, one need not choose sides between coordinate or momentum space. It works in full phase-space, accommodating the uncertainty principle, and it offers unique insights into the classical limit of quantum theory: The variables (observables) in this formulation are c-number functions in phase space instead of operators, with the same interpretation as their classical counterparts, but are composed together in novel algebraic ways.

This volume is a selection of 25 useful papers in the phase-space formulation, with an introductory overview which provides a trail-map to these papers and an extensive bibliography. (Still, the bibliography makes no pretense to exhaustiveness. An up-to-date database on the large literature of the field, with special emphasis on its mathematical and technical aspects, may be found in http://idefix.physik.uni-freiburg.de/∼star/en/download.html .) The overview collects often-used formulas and simple illustrations, suitable for applications to a broad range of physics problems, as well as teaching. It provides supplementary material that may be used for a beginning graduate course in quantum mechanics. D Morrissey is thanked for helpful comments, and T Curtright expresses his obligation to Ms Diaz-Heimer.

Errata and other updates to the book may be found on-line at
http://server.physics.miami.edu/∼curtright/QMPS

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## OVERVIEW OF PHASE-SPACE QUANTIZATION

### 0.1 Introduction

There are at least three logically autonomous alternative paths to quantization. The first is the standard one utilizing operators in Hilbert space, developed by Heisenberg, Schrödinger, Dirac, and others in the 1920s. The second one relies on path integrals, and was conceived by Dirac[Dir33] and constructed by Feynman.

The third one (the bronze medal!) is the phase-space formulation surveyed in this book. It is based on Wigner's (1932) quasi-distribution function[Wig32] and Weyl's (1927) correspondence[Wey27] between quantum-mechanical operators in Hilbert space and ordinary c-number functions in phase space.

The crucial quantum-mechanical composition structure of all such functions, which relies on the $\star$-product, was fully understood by Groenewold (1946)[Gro46], who, together with Moyal (1949)[Moy49], pulled the entire formulation together. Still, insights on interpretation and a full appreciation of its conceptual autonomy took some time to mature with the work of, among others, Takabayasi[Tak54], Baker[Bak58], and Fairlie[Fai64].

This complete formulation is based on the Wigner function (WF), which is a quasi-probability distribution function in phase-space,

$$f(x,p) = \frac{1}{2\pi}\int dy \;\; \psi^*\left(x - \frac{\hbar}{2}y\right) \; e^{-iyp}\psi\left(x + \frac{\hbar}{2}y\right). \tag{1}$$

It is a generating function for all spatial autocorrelation functions of a given quantum-mechanical wave-function $\psi(x)$. More importantly, it is a special representation of the density matrix (in the Weyl correspondence, as detailed in Section 0.12).

Alternatively, in a $2n$-dimensional phase space, it amounts to

$$f(x,p) = \frac{1}{(2\pi\hbar)^n}\int d^n y \;\; \left\langle x + \frac{y}{2}\right| \boldsymbol{\rho} \left| x - \frac{y}{2}\right\rangle \; e^{-ip\cdot y/\hbar}, \tag{2}$$

where $\psi(x) = \langle x|\psi\rangle$ in the density operator $\boldsymbol{\rho}$,

$$\boldsymbol{\rho} \;=\; \int d^n z \int d^n x d^n p \;\; \left|x + \frac{z}{2}\right\rangle f(x,p) \; e^{ip\cdot z/\hbar} \left\langle x - \frac{z}{2}\right|. \tag{3}$$

There are several outstanding reviews on the subject: refs[HOS84,Tak89,Ber80,BJ84,Lit86, deA98,Shi79,Tat83,Coh95,KN91,Kub64,DeG74,KW90, Ber77,Lee95,Dah01,Sch02,DHS00,CZ83,Gad95,HH02, Str57,McD88,Leo97,Sny80,Bal75,BFF78].

Nevertheless, the central conceit of the present overview is that the above input wave-functions may ultimately be bypassed, since the WFs are determined, in principle, as the solutions of suitable functional equations in phase space. Connections to the Hilbert space operator formulation of quantum mechanics may thus be ignored, in principle—even though they are provided in Section 0.12 for pedagogy and confirmation of the formulation's equivalence. One might then envision an imaginary world in which this formulation of quantum mechanics had preceded the conventional Hilbert-space formulation, and its own techniques

and methods had arisen independently, perhaps out of generalizations of classical mechanics and statistical mechanics.

It is not only wave-functions that are missing in this formulation. Beyond the ubiquitous (noncommutative, associative, pseudodifferential) operation, the $\star$-product, which encodes the entire quantum-mechanical action, there are no linear operators. Expectations of observables and transition amplitudes are phase-space integrals of c-number functions, weighted by the WF, as in statistical mechanics.

Consequently, even though the WF is not positive-semidefinite (it can be, and usually is negative in parts of phase-space [Wig32]), the computation of expectations and the associated concepts are evocative of classical probability theory, as emphasized by Moyal. Still, telltale features of quantum mechanics are reflected in the noncommutative multiplication of such c-number phase-space functions through the $\star$-product, in systematic analogy to operator multiplication in Hilbert space.

This formulation of quantum mechanics is useful in describing *quantum* transport processes in phase space, notably in quantum optics[Sch02,Leo97,SM00]; nuclear and particle physics[Bak60,SP81,MM84,CC03,BJY04]; condensed matter [MMP94,DBB02,KKFR89,BP96,Ram04,KL01,JBM03]; the study of semiclassical limits of mesoscopic systems [Imr67,OR57,Sch69,Ber77,KW87,OM95,MS95,MOT98,Vor89,Vo78,Wer95,Ara95,Rob93]; and the transition to classical statistical mechanics [VMdG61,JD99,Fre87,BD98,Dek77,Raj83,HY96,CV98,SM00,FLM98,FZ01,Zal03,CKTM07].

Since observables are expressed by virtually common variables in both their quantum and classical configurations, this formulation is the natural language in which to investigate 'quantum chaos' and decoherence[Ber77,JN90,Zu91,ZP94,BC99,KZZ02,KJ99,FBA96,Kol96,GH93,CL03,BTU93,Mon94,HP03,OC03] (of utility in, e.g., quantum computing[BHP02,MPS02]); and provides crucial intuition in quantum mechanical interference problems[Wis97], molecular Talbot-Lau interferometry[NH08], probability flows as negative probability backflows[BM94,FMS00,BV90], and measurements of atomic systems[Smi93,Dun95,Lei96,KPM97,Lvo01,JS02,BHS02,Ber02,Cas91].

The intriguing mathematical structure of the formulation is of relevance to Lie Algebras[FFZ89]; martingales in turbulence[Fan03]; and string field theory[BKM03]. It has recently been retrofitted into M-theory and quantum field theory advances linked to noncommutative geometry[SW99,Fil96] (for reviews, see [Cas00,Har01,DN01,HS02]), and matrix models[Tay01,KS02]; these apply spacetime uncertainty principles[Pei33,Yo89,JY98,SST00] reliant on the $\star$-product. (Transverse spatial dimensions act formally as momenta, and, analogously to quantum mechanics, their uncertainty is increased or decreased inversely to the uncertainty of a given direction.)

As a significant aside, the WF has extensive practical applications in signal processing, filtering, and engineering (time-frequency analysis), since time and frequency constitute a

pair of Fourier-conjugate variables just like the $\mathfrak{x}$ and $\mathfrak{p}$ pair of phase space[a].

For simplicity, the formulation will be mostly illustrated for one coordinate and its conjugate momentum, but generalization to arbitrary-sized phase spaces is straightforward[Bal75,DM86], including infinite-dimensional ones, namely scalar field theory[Dit90,Les84,Na97,CZ99,CPP01,MM94]: the respective WFs are simple products of single-particle WFs.

## 0.2 The Wigner Function

As already indicated, the quasi-probability measure in phase space is the WF,

$$f(x,p) = \frac{1}{2\pi} \int dy \; \psi^* \left( x - \frac{\hbar}{2} y \right) \; e^{-iyp} \; \psi \left( x + \frac{\hbar}{2} y \right). \tag{4}$$

It is obviously normalized, $\int dpdx f(x,p) = 1$. In the classical limit, $\hbar \to 0$, it would reduce to the probability density in coordinate space $x$, usually highly localized, multiplied by $\delta$-functions in momentum: the classical limit is "spiky" and certain!

This expression has more $x-p$ symmetry than is apparent, as Fourier transformation to momentum-space wave-functions yields a completely symmetric expression with the roles of $x$ and $p$ reversed, and, upon rescaling of the arguments $x$ and $p$, a symmetric classical limit.

The WF is also manifestly real[b]. It is also constrained by the Cauchy-Schwarz inequality to be bounded, $-\frac{2}{\hbar} \leq f(x,p) \leq \frac{2}{\hbar}$. Again, this bound disappears in the spiky classical limit. Thus, this quantum-mechanical bound precludes a WF which is a perfectly localized delta function in phase space—the uncertainty principle.

Respectively, $p$- or $x$-projection leads to marginal probability densities: a spacelike shadow $\int dp \; f(x,p) = \rho(x)$, *or else* a momentum-space shadow $\int dx f(x,p) = \sigma(p)$. Either is a bona-fide probability density, being positive semidefinite. But neither can be conditioned on the other, as the uncertainty principle is fighting back: The WF $f(x,p)$ itself can, and most often is negative in some small areas of phase-space[Wig32,HOS84,MLD86], as is illustrated below, a hallmark of QM interference in this language. Such negative features thus serve to monitor quantum coherence; and their attenuation, respectively, its loss. (In fact, the only pure state WF which is non-negative is the Gaussian[Hud74], a state of maximum entropy[Raj83].)

[a] Thus, time varying signals are best represented in a WF as time varying spectrograms, analogously to a music score, i.e. the changing distribution of frequencies is monitored in time[deB67,BBL80,Wok97,QC96,MH97,Coh95,Gro01,Fla99]: even though the description is constrained and redundant, it gives an intuitive picture of the signal that a mere time profile or frequency spectrogram fails to convey. Applications abound[CGB91,Lou96,MH97] in bioengineering, acoustics, speech analysis, vision processing, turbulence microstructure analysis, radar imaging, seismic data analysis, and the monitoring of internal combustion engine-knocking, failing helicopter component vibrations, and so on.

[b] In one space dimension, by virtue of non-degeneracy, $\psi$ has the same effect as $\psi^*$, and $f$ turns out to be $p$-even; but this is not a property used here.

The counter-intuitive "negative probability" aspects of this quasi-probability distribution have been explored and interpreted [Bar45,Fey87,BM94,MLD86] (for a popular review, see [LPM98]), and negative probability flows amount to legitimate probability backflows in interesting settings [BM94]. Nevertheless, the WF for atomic systems can still be measured in the laboratory, albeit indirectly, and reconstructed [Smi93,Dun95,Lei96,KPM97,Lvo01,Lut96,BAD96,BHS02,Ber02,BRWK99,Vog89].

Smoothing $f$ by a filter of size larger than $\hbar$ (e.g., convolving with a phase-space Gaussian) results in a positive-semidefinite function, i.e. it may be thought to have been smeared or blurred to a classical[c] distribution[deB67,Car76,Ste80,OW81,Raj83].

It is thus evident that patches in $f$ of uniformly negative value cannot be much larger than a few $\hbar$, since, then, smoothing by an $\hbar$-filter would fail to obliterate them. That is, negative patches are *small*, a microscopic phenomenon, in general, in some sense shielded by the uncertainty principle. Monitoring negative WF features and their attenuation in time affords a measure of decoherence and drift towards a classical state[KJ99].

Among real functions, the WFs comprise a rather small, highly constrained, set. When is a real function $f(x,p)$ a bona-fide Wigner function of the form (4)? Evidently, when its Fourier transform (the cross-spectral density) "left-right" factorizes,

$$\tilde{f}(x,y) = \int dp\ e^{ipy} f(x,p) = g_L^*(x-\hbar y/2)\ g_R(x+\hbar y/2)\ . \tag{5}$$

That is,

$$\frac{\partial^2\ \ln \tilde{f}}{\partial(x-\hbar y/2)\ \partial(x+\hbar y/2)} = 0\ , \tag{6}$$

so, for real $f$, $\quad g_L = g_R$.

Nevertheless, as indicated, the WF *is* a distribution function, after all: it provides the integration measure in phase space to yield expectation values of observables from corresponding phase-space c-number functions. Such functions are often familiar classical quantities; but, in general, they are uniquely associated to suitably ordered operators through *Weyl's correspondence rule*[Wey27]. Given an operator (in gothic script) ordered in this prescription,

$$\mathfrak{G}(\mathfrak{x},\mathfrak{p}) = \frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp\ g(x,p)\exp(i\tau(\mathfrak{p}-p)+i\sigma(\mathfrak{x}-x))\ , \tag{7}$$

[c]This one is called the Husimi distribution[Tak89,TA99], and sometimes information scientists examine it on account of its non-negative feature. Nevertheless, it comes with a substantially heavy price, as it needs to be "dressed" back to the WF, for all practical purposes, when expectation values are computed with it; i.e., unlike the WF, it does not serve as an immediate quasi-probability distribution with no further measure (see Section 0.13). The negative feature of the WF is, in the last analysis, an asset, and not a liability, and provides an efficient description of "beats"[BBL80,Wok97,QC96,MH97,Coh95], cf. Fig. 1. If, instead, strictly inequivalent (improper) expectation values were taken with the Husimi distribution *without* the requisite dressing of Section 0.13, i.e. as though it were a bona-fide probability distribution, such expectation values would reflect loss of quantum information: they would represent classically smeared observables[WO87].

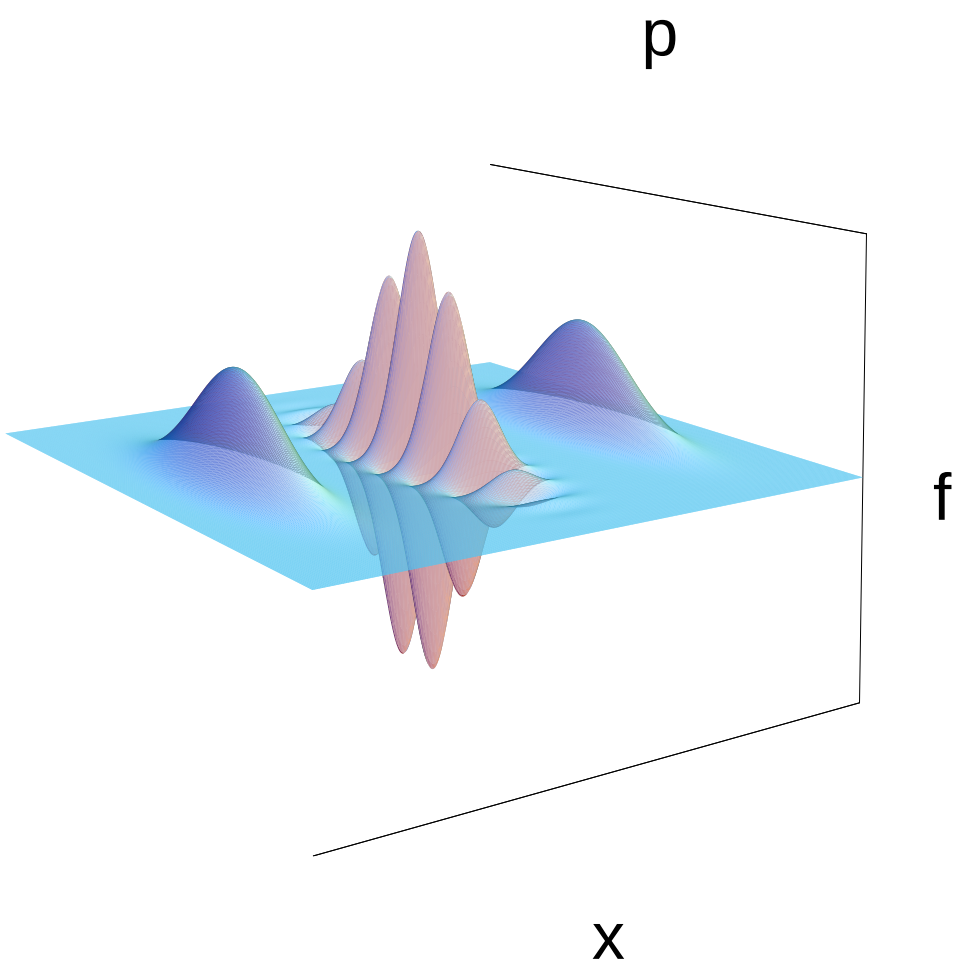


Figure 1. Wigner function of a pair of Gaussian wavepackets centered at $x = \pm a$: $f(x,p;a) = \exp(-(x^2+p^2))(\exp(-a^2)\cosh(2ax) + \cos(2pa))/(\pi(1+e^{-a^2}))$. (For simplicity, $\hbar = 1$ here. The corresponding wave-function is $\psi(x;a) = \left(\exp\left(-(x+a)^2/2\right) + \exp\left(-(x-a)^2/2\right)\right)/(\pi^{1/4}\sqrt{2+2e^{-a^2}})$.) Here, $a = 6$ is chosen, quite larger than the width of the Gaussians. Note the phase-space interference structure ("beats") with negative values in the $x$ region between the two packets where there is no wave-function support—hence vanishing probability for the presence of the particle. The oscillation frequency in the $p$-direction is $a/\pi$. Thus, it increases with growing separation $a$, ultimately smearing away the interference structure.

the corresponding phase-space function $g(x,p)$ (the *Weyl kernel function* or the *Wigner transform* of the operator) is obtained by

$$\mathfrak{p} \mapsto p, \qquad \mathfrak{x} \mapsto x \ . \tag{8}$$

That operator's expectation value is then given by a "phase-space average" [Gro46,Moy49],

$$\langle \mathfrak{G} \rangle = \int dxdp \ f(x,p) \ g(x,p). \tag{9}$$

The kernel function $g(x,p)$ is often the unmodified classical observable expression, such as a conventional Hamiltonian, $H = p^2/2m + V(x)$, i.e. the transition from classical mechanics is straightforward ("quantization").

However, the kernel function contains $\hbar$ corrections when there are quantum-mechanical ordering ambiguities in the observables, such as in the kernel of the square of the angular momentum, $\mathfrak{L} \cdot \mathfrak{L}$. This one contains a term $-3\hbar^2/2$ introduced by the Weyl ordering[She59,DS82,DS02], beyond the mere classical expression ($L^2$)—and accounts for the nontrivial angular momentum of the ground-state Bohr orbit.

In such cases (including momentum-dependent potentials), even nontrivial $O(\hbar)$ quantum corrections in the kernel functions (which characterize different operator orderings) can be produced efficiently without direct, cumbersome consideration of operators[CZ02,Hie84]. More detailed discussion of the Weyl and alternate correspondence maps is provided in Sections 0.12 and 0.13.

In this sense, expectation values of the physical observables specified by kernel functions $g(x,p)$ are computed through integration with the WF, $f(x,p)$, in close analogy to classical probability theory, except for the non-positive-definiteness of the distribution function. This operation corresponds to tracing an operator with the density matrix (cf. Section 0.12).

## 0.3 Solving for the Wigner Function

Given a specification of observables, the next step is to find the relevant WF for a given Hamiltonian. Can this be done without solving for the Schrödinger wavefunctions $\psi$, i.e. not using Schrödinger's equation directly? Indeed, the functional equations which $f$ satisfies completely determine it.

Firstly, its dynamical evolution is specified by Moyal's equation. This is the extension of Liouville's theorem of classical mechanics, for a classical Hamiltonian $H(x,p)$, namely $\partial_t f + \{f, H\} = 0$, to quantum mechanics, in this language[Wig32,Moy49]:

$$\frac{\partial f}{\partial t} = \frac{H \star f - f \star H}{i\hbar} \equiv \{\!\{H, f\}\!\} \ , \tag{10}$$

where the $\star$-product[Gro46] is

$$\star \equiv e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)} \ . \tag{11}$$

The right-hand side of (10) is dubbed the "Moyal Bracket" (MB), and the quantum commutator is its Weyl-correspondent (its Weyl transform). It is the essentially unique one-parameter ($\hbar$) associative deformation (expansion) of the Poisson Brackets (PB) of classical mechanics[Vey75,BFF78,FLS76,Ar83,Fle90,deW83,BCG97,TD97]. Expansion in $\hbar$ around 0 reveals that it consists of the Poisson Bracket corrected by terms $O(\hbar)$.

The equation (10) also evokes Heisenberg's equation of motion for operators, except $H$ and $f$ here are ordinary "classical" functions, and it is the $\star$-product which now enforces noncommutativity. This language, then, makes the link between quantum commutators and Poisson Brackets more transparent.

Since the $\star$-product involves exponentials of derivative operators, it may be evaluated in practice through translation of function arguments ("Bopp shifts"),

$$f(x,p)\star g(x,p) = f\left(x+\frac{i\hbar}{2}\overrightarrow{\partial}_p,\ p-\frac{i\hbar}{2}\overrightarrow{\partial}_x\right)\ g(x,p). \tag{12}$$

The equivalent Fourier representation of the $\star$-product is[Neu31,Bak58]

$$\begin{aligned} f\star g = \frac{1}{\hbar^2\pi^2}\int dp'dp''dx'dx''\ f(x',p')\ g(x'',p'') \\ \times\exp\left(\frac{-2i}{\hbar}\left(p(x'-x'')+p'(x''-x)+p''(x-x')\right)\right). \end{aligned} \tag{13}$$

An alternate integral representation of this product is[HOS84]

$$f\star g = (\hbar\pi)^{-2}\int dp'dp''dx'dx''\ f(x+x',p+p')\ g(x+x'',p+p'')\ \exp\left(\frac{2i}{\hbar}\left(x'p''-x''p'\right)\right), \tag{14}$$

which readily displays noncommutativity and associativity.

$\star$-multiplication of c-number phase-space functions is in complete isomorphism to Hilbert-space operator multiplication[Gro46] of the respective Weyl transforms,

$$\mathfrak{A}(\mathfrak{x},\mathfrak{p})\ \mathfrak{B}(\mathfrak{x},\mathfrak{p}) = \frac{1}{(2\pi)^2}\int d\tau d\sigma dxdp\ (a\star b)\ \exp(i\tau(\mathfrak{p}-p)+i\sigma(\mathfrak{x}-x)). \tag{15}$$

The cyclic phase-space trace is directly seen in the representation (14) to reduce to a plain product, if there is *only one* $\star$ involved,

$$\int dpdx\ f\star g = \int dpdx\ fg = \int dpdx\ g\star f. \tag{16}$$

Moyal's equation is necessary, but does not suffice to specify the WF for a system. In the conventional formulation of quantum mechanics, systematic solution of time-dependent equations is usually predicated on the spectrum of stationary ones. Time-independent pure-state Wigner functions $\star$-commute with $H$, but clearly not every function $\star$-commuting with $H$ can be a bona-fide WF (e.g., any $\star$-function of $H$ will $\star$-commute with $H$).

Static WFs obey more powerful functional $\star$-genvalue equations[Fai64] (also see [Kun67,Coh76,Dah83]),

$$H(x,p)\star f(x,p) = H\left(x+\frac{i\hbar}{2}\overrightarrow{\partial}_p\ ,\ p-\frac{i\hbar}{2}\overrightarrow{\partial}_x\right)\ f(x,p)$$
$$= f(x,p)\star H(x,p) = E\ f(x,p)\ , \tag{17}$$

where $E$ is the energy eigenvalue of $\mathfrak{H}\psi = E\psi$ in Hilbert space. These amount to a complete characterization of the WFs[CFZ98]. (NB. Observe the $\hbar\to 0$ transition to the classical limit.)

**Lemma 0.1** For real functions $f(x,p)$, the Wigner form (4) for pure static eigenstates is equivalent to compliance with the $\star$-genvalue equations (17) ($\Re$ and $\Im$ parts).

**Proof**

$$\begin{aligned}
&H(x,p)\star f(x,p) = \\
&= \frac{1}{2\pi}\left((p-i\frac{\hbar}{2}\overrightarrow{\partial}_x)^2/2m+V(x)\right)\int dy\ e^{-iy(p+i\frac{\hbar}{2}\overleftarrow{\partial}_x)}\psi^*(x-\frac{\hbar}{2}y)\ \psi(x+\frac{\hbar}{2}y)\\
&= \frac{1}{2\pi}\int dy\ \left((p-i\frac{\hbar}{2}\overrightarrow{\partial}_x)^2/2m+V(x+\frac{\hbar}{2}y)\right)e^{-iyp}\psi^*(x-\frac{\hbar}{2}y)\ \psi(x+\frac{\hbar}{2}y)\\
&= \frac{1}{2\pi}\int dy\ e^{-iyp}\left((i\overrightarrow{\partial}_y+i\frac{\hbar}{2}\overrightarrow{\partial}_x)^2/2m+V(x+\frac{\hbar}{2}y)\right)\psi^*(x-\frac{\hbar}{2}y)\ \psi(x+\frac{\hbar}{2}y)\\
&= \frac{1}{2\pi}\int dy\ e^{-iyp}\psi^*(x-\frac{\hbar}{2}y)\ E\ \psi(x+\frac{\hbar}{2}y)\\
&= E\ f(x,p).
\end{aligned} \tag{18}$$

Action of the effective differential operators on $\psi^*$ turns out to be null.

Symmetrically,

$$\begin{aligned}
&f\star H = \\
&\quad = \frac{1}{2\pi}\int dy\ e^{-iyp}\left(-\frac{1}{2m}(\overrightarrow{\partial}_y-\frac{\hbar}{2}\overrightarrow{\partial}_x)^2+V(x-\frac{\hbar}{2}y)\right)\psi^*(x-\frac{\hbar}{2}y)\ \psi(x+\frac{\hbar}{2}y)\\
&\quad = E\ f(x,p),
\end{aligned} \tag{19}$$

where the action on $\psi$ is now trivial.

Conversely, the pair of $\star$-eigenvalue equations dictate, for $f(x,p)=\int dy\ e^{-iyp}\tilde{f}(x,y)$ ,

$$\int dy\ e^{-iyp}\left(-\frac{1}{2m}(\overrightarrow{\partial}_y\pm\frac{\hbar}{2}\overrightarrow{\partial}_x)^2+V(x\pm\frac{\hbar}{2}y)-E\right)\tilde{f}(x,y)=0. \tag{20}$$

Hence, Real solutions of (17) must be of the form
$f=\int dy\ e^{-iyp}\psi^*(x-\frac{\hbar}{2}y)\psi(x+\frac{\hbar}{2}y)/2\pi$, such that $\mathfrak{H}\psi = E\psi$. □

The eqs (17) lead to spectral properties for WFs[Fai64,CFZ98], as in the Hilbert space formulation. For instance, projective orthogonality of the $\star$-genfunctions follows from associativity, which allows evaluation in two alternate groupings:

$$f\star H\star g = E_f\ f\star g = E_g\ f\star g. \tag{21}$$

Thus, for $E_g \neq E_f$, it is necessary that

$$f \star g = 0. \tag{22}$$

Moreover, precluding degeneracy (which can be treated separately), choosing $f = g$ above yields,

$$f \star H \star f = E_f \; f \star f = H \star f \star f, \tag{23}$$

and hence $f \star f$ must be the stargenfunction in question,

$$f \star f \propto f. \tag{24}$$

Pure state $f$s then $\star$-project onto their space. In general, it can be shown[Tak54,CFZ98] that, for a pure state,

$$f_a \star f_b = \frac{1}{h} \; \delta_{a,b} \; f_a \; . \tag{25}$$

The normalization matters[Tak54]: despite linearity of the equations, it prevents naive superposition of solutions. (Quantum mechanical interference works differently here, in comportance with conventional density-matrix formalism.)

By virtue of (16), for different $\star$-genfunctions, the above dictates that

$$\int dpdx \; fg = 0. \tag{26}$$

Consequently, unless there is zero overlap for all such WFs, at least one of the two must go negative someplace to offset the positive overlap [HOS84,Coh95]—an illustration of the feature of negative values. This feature is *an asset and not a liability.*

Further note that integrating (17) yields the expectation of the energy,

$$\int H(x,p) f(x,p) \; dxdp = E \int f \; dxdp = E. \tag{27}$$

N.B. Likewise[d], integrating the above projective condition yields

$$\int dxdp \; f^2 = \frac{1}{h} \; , \tag{28}$$

that is the overlap increases to a divergent result in the classical limit, as the WFs grow increasingly spiky.

[d]This discussion applies to proper WFs, corresponding to pure states' density matrices. E.g., a sum of two WFs is not a pure state in general, and does not satisfy the condition (6). For such generalizations, the *impurity* is[Gro46] $1 - h\langle f \rangle = \int dxdp \; (f - hf^2) \geq 0$, where the inequality is only saturated into an equality for a pure state. For instance, for $w \equiv (f_a + f_b)/2$ with $f_a \star f_b = 0$, the impurity is nonvanishing, $\int dxdp \; (w - hw^2) = 1/2$. A pure state affords a maximum of information, while the impurity is a measure of lack of information[Fan57,Tak54]—it is the dominant term in the expansion of the quantum entropy around a pure state[Bra94].

## 0.4 The Uncertainty Principle

In classical (non-negative) probability distribution theory, expectation values of non-negative functions are likewise non-negative, and thus result in standard constraint inequalities for the constituent pieces of such functions, such as, e.g., moments of the variables.

But it was just discussed that, for WFs which go negative for an arbitrary function $g$, the expectation $\langle |g|^2 \rangle$ need not be $\geq 0$. This can be easily illustrated by choosing the support of $g$ to lie mostly in those (small) regions of phase-space where the WF $f$ is negative.

Still, such constraints are not lost for WFs. It turns out they are replaced by

**Lemma 0.2**

$$\langle g^* \star g \rangle \geq 0. \tag{29}$$

In Hilbert space operator formalism, this relation would correspond to the positivity of the norm. This expression is non-negative because it involves a real non-negative integrand for a pure state WF satisfying the above projective condition[e],

$$\int dpdx(g^* \star g)f = h \int dxdp(g^* \star g)(f \star f) = h \int dxdp(f \star g^*) \star (g \star f) = h \int dxdp|g \star f|^2. \tag{30}$$

□

To produce Heisenberg's uncertainty relation[CZ01], one only need choose

$$g = a + bx + cp, \tag{31}$$

for arbitrary complex coefficients $a, b, c$. The resulting positive semi-definite quadratic form is then

$$a^*a + b^*b\langle x \star x \rangle + c^*c\langle p \star p \rangle + (a^*b + b^*a)\langle x \rangle + (a^*c + c^*a)\langle p \rangle + c^*b\langle p \star x \rangle + b^*c\langle x \star p \rangle \geq 0 \ , \tag{32}$$

for any $a, b, c$. The eigenvalues of the corresponding matrix are then non-negative, and thus so must be its determinant. Given

$$x \star x = x^2, \qquad p \star p = p^2, \qquad p \star x = px - i\hbar/2 \ , \qquad x \star p = px + i\hbar/2 \ , \tag{33}$$

and the usual

$$(\Delta x)^2 \equiv \langle (x - \langle x \rangle)^2 \rangle, \qquad\qquad (\Delta p)^2 \equiv \langle (p - \langle p \rangle)^2 \rangle, \tag{34}$$

this condition on the $3 \times 3$ matrix determinant amounts to

$$(\Delta x)^2 \ (\Delta p)^2 \geq \hbar^2/4 + \Big( \langle (x - \langle x \rangle)(p - \langle p \rangle) \rangle \Big)^2, \tag{35}$$

and hence

$$\Delta x \ \Delta p \geq \frac{\hbar}{2} \ . \tag{36}$$

The $\hbar$ entered into the moments' constraint through the action of the $\star$-product [CZ01].

[e]Similarly, if $f_1$ and $f_2$ are pure state WFs, the transition probability ($|\int dx \psi_1^*(x)\psi_2(x)|^2$) between the respective states is also non-negative[OW81], manifestly by the same argument[CZ01], providing for a non-negative phase-space overlap, $\int dpdx f_1 f_2 = (2\pi\hbar)^2 \int dxdp \ |f_1 \star f_2|^2 \geq 0$.

More general choices of $g$ likewise lead to diverse expectations' inequalities in phase space; e.g., in 6-dimensional phase space, the uncertainty for $g = a + bL_x + cL_y$ requires $l(l+1) \geq m(m+1)$, and hence $l \geq m$, and so forth[CZ01,CZ02]. For a more extensive formal discussion of moments, cf. [NO86].

## 0.5 Ehrenfest's Theorem

Moyal's equation (10),

$$\frac{\partial f}{\partial t} = \{\!\{H, f\}\!\} \ , \tag{37}$$

serves to prove Ehrenfest's theorem for expectation values.

For any phase-space function $k(x,p)$ with no explicit time-dependence,

$$\begin{aligned} \frac{d\langle k\rangle}{dt} &= \int dxdp \ \frac{\partial f}{\partial t} k \\ &= \frac{1}{i\hbar} \int dxdp \ (H \star f - f \star H) \star k \\ &= \int dxdp \ f \{\!\{k, H\}\!\} = \langle \{\!\{k, H\}\!\} \rangle \ . \end{aligned} \tag{38}$$

(Any convective time-dependence, $\int dxdp \ (\dot{x}\partial_x \ (fk) + \dot{p} \ \partial_p(fk))$, amounts to an ignorable surface term, $\int dxdp \ (\partial_x(\dot{x}fk) + \partial_p(\dot{p}fk))$, by the $x, p$ equations of motion.)

Note the ostensible sign difference between the correspondent to Heisenberg's equation,

$$\frac{dk}{dt} = \{\!\{k, H\}\!\} \ , \tag{39}$$

and Moyal's equation above. The $x, p$ equations of motion in such a Heisenberg picture, then, reduce to the classical ones of Hamilton, $\dot{x} = \partial_p H$, $\dot{p} = -\partial_x H$.

Moyal [Moy49] stressed that his eponymous quantum evolution equation (10) contrasts to Liouville's theorem for classical phase-space densities,

$$\frac{df_{cl}}{dt} = \frac{\partial f_{cl}}{\partial t} + \dot{x} \ \partial_x f_{cl} + \dot{p} \ \partial_p f_{cl} = 0 \ . \tag{40}$$

Specifically, unlike its classical counterpart, in general, $f$ does not flow like an incompressible fluid in phase space.

For an arbitrary region $\Omega$ about some representative point in phase space,

$$\frac{d}{dt}\int_\Omega dxdp \ f = \int_\Omega dxdp \left( \frac{\partial f}{\partial t} + \partial_x(\dot{x}f) + \partial_p(\dot{p}f) \right) = \int_\Omega dxdp \ (\{\!\{H, f\}\!\} - \{H, f\}) \neq 0 \ . \tag{41}$$

That is, the phase-space region does not conserve in time the number of points swarming about the representative point: points diffuse away, in general, without maintaining the density of the quantum quasi-probability fluid; and, conversely, they are not prevented

from coming together, in contrast to deterministic flow behavior. Still, for infinite $\Omega$ encompassing the entire phase space, both surface terms above vanish to yield a time-invariant normalization for the WF.

The $O(\hbar^2)$ higher momentum derivatives of the WF present in the MB (but absent in the PB—higher space derivatives probing nonlinearity in the potential) modify the Liouville flow into characteristic quantum configurations[KZZ02,FBA96,ZP94].

## 0.6 Illustration: the Harmonic Oscillator

To illustrate the formalism on a simple prototype problem, one may look at the harmonic oscillator. In the spirit of this picture, in fact, one can eschew solving the Schrödinger problem and plugging the wavefunctions into (4); instead, for $H = (p^2 + x^2)/2$ (with $m = 1$, $\omega = 1$), one may solve (17) directly,

$$\left(\left(x + \frac{i\hbar}{2}\partial_p\right)^2 + \left(p - \frac{i\hbar}{2}\partial_x\right)^2 - 2E\right) f(x,p) = 0. \tag{42}$$

For this Hamiltonian, then, the equation has collapsed to two simple Partial Differential Equations.

The first one, the Imaginary part,

$$(x\partial_p - p\partial_x)f = 0, \tag{43}$$

restricts $f$ to depend on only one variable, the scalar in phase space,

$$z \equiv \frac{4}{\hbar}H = \frac{2}{\hbar}(x^2 + p^2) \ .$$

Thus the second one, the Real part, is a simple Ordinary Differential Equation,

$$\left(\frac{z}{4} - z\partial_z^2 - \partial_z - \frac{E}{\hbar}\right) f(z) = 0. \tag{44}$$

Setting $f(z) = \exp(-z/2)L(z)$ yields Laguerre's equation,

$$\left(z\partial_z^2 + (1-z)\partial_z + \frac{E}{\hbar} - \frac{1}{2}\right) L(z) = 0. \tag{45}$$

It is solved by Laguerre polynomials,

$$L_n = \frac{1}{n!}e^z \ \partial_z^n(e^{-z}z^n) \ , \tag{46}$$

for $n = E/\hbar - 1/2 = 0, 1, 2, ...$, so the $\star$-gen-Wigner-functions are[Gro46]

$$f_n = \frac{(-1)^n}{\pi\hbar} \ e^{-2H/\hbar} \ L_n\left(\frac{4H}{\hbar}\right) \ ;$$

$$L_0 = 1, \quad L_1 = 1 - \frac{4H}{\hbar}, \quad L_2 = \frac{8H^2}{\hbar^2} - \frac{8H}{\hbar} + 1 \ , ... \tag{47}$$

But for the Gaussian ground state, they all have zeros and go negative.

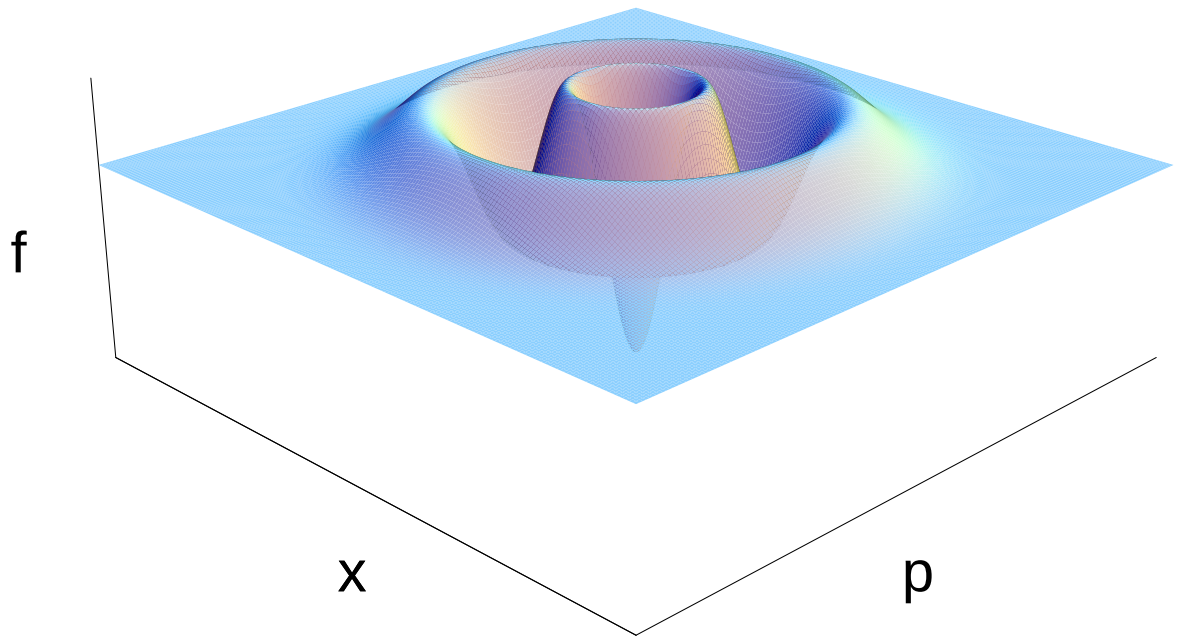


Figure 2. The oscillator WF for the 3rd excited state. Note the negative values.

These functions become spiky in the classical limit $\hbar \to 0$; e.g., the ground state Gaussian $f_0$ goes to a $\delta$-function. Their sum provides a resolution of the identity[Moy49],

$$\sum_n f_n = \frac{1}{h} \ . \tag{48}$$

Note that the quantum energy variance is

$$\langle H \star H \rangle - \langle H \rangle^2 = (\langle H^2 \rangle - \langle H \rangle^2) - \frac{\hbar^2}{4} \ ,$$

vanishing for all $\star$-genstates; while the naive star-less variance on the right-hand side is thus larger than that, $\hbar^2/4$, and would groundlessly suggest broader dispersion.

(For the rest of this section, set $\hbar = 1$, for algebraic simplicity.)

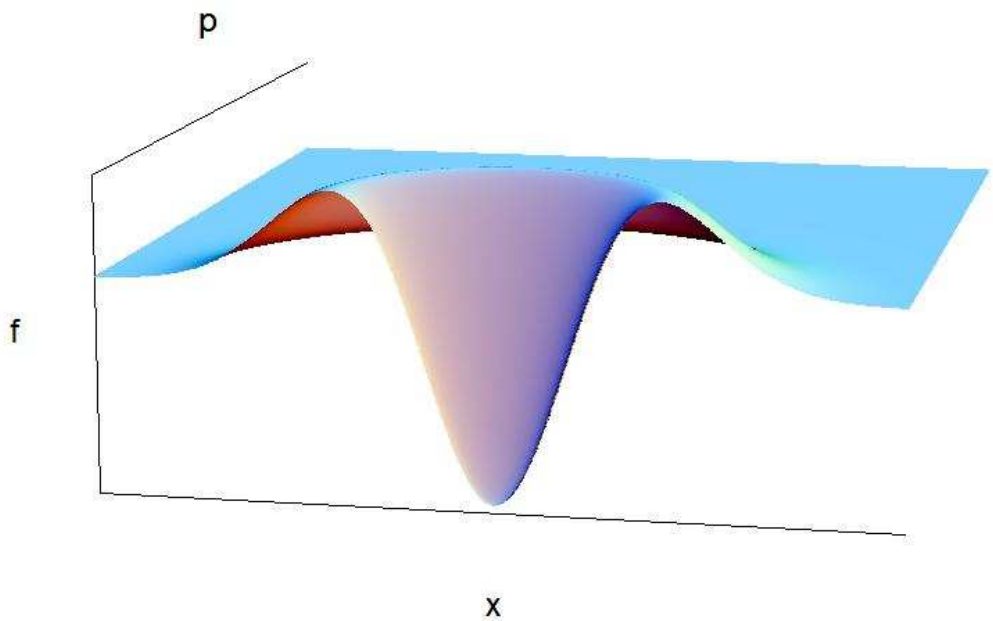


Figure 3. Section of the WF for the first excited state. Note the negative values.

Dirac's Hamiltonian factorization method for the alternate algebraic solution of the same problem carries through intact, with $\star$-multiplication supplanting operator multiplication. That is to say,

$$H = \frac{1}{2}(x - ip) \star (x + ip) + \frac{1}{2} \ . \tag{49}$$

This motivates definition of raising and lowering functions (not operators)

$$a \equiv \frac{1}{\sqrt{2}}(x + ip), \qquad a^\dagger \equiv a^* = \frac{1}{\sqrt{2}}(x - ip), \tag{50}$$

where

$$a \star a^\dagger - a^\dagger \star a = 1 \ . \tag{51}$$

The annihilation functions $\star$-annihilate the $\star$-Fock vacuum,

$$a \star f_0 = \frac{1}{\sqrt{2}}(x + ip) \star e^{-(x^2+p^2)} = 0 \ . \tag{52}$$

Thus, the associativity of the $\star$-product permits the customary ladder spectrum generation[CFZ98]. The $\star$-genstates for $H \star f = f \star H$ are then

$$f_n = \frac{1}{n!}(a^\dagger \star)^n \ f_0 \ (\star a)^n \ . \tag{53}$$

They are manifestly real, like the Gaussian ground state, and left-right symmetric; it is easy to see they are $\star$-orthogonal for different eigenvalues. Likewise, they can be seen by the evident algebraic normal ordering to project to themselves, since the Gaussian ground state does, $f_0 \star f_0 = f_0/h$.

The corresponding coherent state WFs [HKN88,Sch88,CUZ01,Har01,DG80] are likewise analogous to the conventional formulation, amounting to this ground state with a displacement in the phase-space origin.

This type of analysis carries over well to a broader class of problems[CFZ98] with "essentially isospectral" pairs of partner potentials, connected with each other through Darboux transformations relying on Witten superpotentials $W$ (cf. the Pöschl-Teller potential[Ant01]). It closely parallels the standard differential operator structure of the recursive technique. That is, the pairs of related potentials and corresponding $\star$-genstate Wigner functions are constructed recursively[CFZ98] through ladder operations analogous to the algebraic method outlined above for the oscillator.

Beyond such recursive potentials, examples of further simple systems where the $\star$-genvalue equations can be solved on first principles include the linear potential [GM80,CFZ98,TZM96], the exponential interaction Liouville potentials, and their supersymmetric Morse generalizations [CFZ98]. (Also see [Fra00,DS82,CH86,HL99,KL94]).

Further systems may be handled through the Chebyshev-polynomial numerical techniques of ref [HMS98].

First principles phase-space solution of the Hydrogen atom is less than straightforward and complete. The reader is referred to [BFF78,Bon84,DS82,CH87] for significant partial results.

Algebraic methods of generating spectra of quantum integrable models are described in ref [CZ02].

## 0.7 Time Evolution

Moyal's equation (10) is formally solved by virtue of associative combinatoric operations completely analogous to Hilbert space quantum mechanics, through definition of a $\star$-unitary

evolution operator, a "$\star$-exponential"[Imr67,GLS68,BFF78]

$$U_\star(x,p;t) = e_\star^{itH/\hbar} \equiv 1 + (it/\hbar)H(x,p) + \frac{(it/\hbar)^2}{2!} H \star H + \frac{(it/\hbar)^3}{3!} H \star H \star H + ..., \quad (54)$$

for arbitrary Hamiltonians. The solution to Moyal's equation, given the WF at $t = 0$, then, is

$$f(x,p;t) = U_\star^{-1}(x,p;t) \star f(x,p;0) \star U_\star(x,p;t). \quad (55)$$

In general, just like any $\star$-function of $H$, the $\star$-exponential (54) resolves spectrally [Bon84],

$$\exp_\star\left(\frac{it}{\hbar}H\right) = \exp_\star\left(\frac{it}{\hbar}H\right) \star 1 = \exp_\star\left(\frac{it}{\hbar}H\right) \star 2\pi\hbar \sum_n f_n = 2\pi\hbar \sum_n e^{itE_n/\hbar} f_n \ . \quad (56)$$

(Of course, for $t = 0$, the obvious identity resolution is recovered.) In turn, any particular $\star$-genfunction is projected out formally by

$$\int dt \exp_\star\left(\frac{it}{\hbar}(H - E_m)\right) = (2\pi\hbar)^2 \sum_n \delta(E_n - E_m) f_n \ \propto f_m \ , \quad (57)$$

which is manifestly seen to be a $\star$-function.

For oscillator $\star$-genfunctions, the $\star$-exponential (56) is directly seen to sum to

$$\exp_\star\left(\frac{itH}{\hbar}\right) = \left(\cos(\frac{t}{2})\right)^{-1} \exp\left(\frac{2i}{\hbar} H \tan(\frac{t}{2})\right), \quad (58)$$

which is, to say, just a Gaussian[BM49,Imr67,BFF78] in phase space[f].

N.B. This $\star$-exponential (56) for the oscillator may be evaluated alternatively[BFF78] without explicit knowledge of the $\star$-genfunctions $f_n$ summed above. Instead, for (54), $U(H,t) \equiv \exp_\star(itH/\hbar)$, Laguerre's equation emerges again,

$$\partial_t U = \frac{i}{\hbar} H \star U = i\left(\frac{H}{\hbar} - \frac{\hbar}{4}(\partial_H + H\partial_H^2)\right) U \ ,$$

and is solved by (58). One may then read off the $f_n$s in (56) as the Fourier coefficients of $U$.

For the variables $x$ and $p$, in the Heisenberg picture, the evolution equations collapse to mere *classical* trajectories for the oscillator,

$$\frac{dx}{dt} = \frac{x \star H - H \star x}{i\hbar} = \partial_p H = p \ , \quad (59)$$

$$\frac{dp}{dt} = \frac{p \star H - H \star p}{i\hbar} = -\partial_x H = -x \ , \quad (60)$$

[f] As an application, note that the celebrated hyperbolic tangent $\star$-composition law of Gaussians follows trivially, since these amount to $\star$-exponentials with additive time intervals, $\exp_\star(tf) \star \exp_\star(Tf) = \exp_\star((t+T)f)$,[BFF78]. That is,

$$\exp\left(-\frac{a}{\hbar}(x^2+p^2)\right) \ \star \ \exp\left(-\frac{b}{\hbar}(x^2+p^2)\right) = \frac{1}{1+ab} \exp\left(-\frac{a+b}{\hbar(1+ab)}(x^2+p^2)\right).$$

where the concluding member of these two equations only hold for the oscillator. Thus, for the oscillator,

$$x(t) = x\cos t + p\sin t, \qquad p(t) = p\cos t - x\sin t. \tag{61}$$

As a consequence, for the oscillator, the functional form of the Wigner function is preserved along classical phase-space trajectories[Gro46],

$$f(x,p;t) = f(x\cos t - p\sin t, p\cos t + x\sin t; 0). \tag{62}$$

*Any* oscillator WF configuration rotates uniformly on the phase plane around the origin, essentially classically, (cf. Fig. 4), even though it provides a complete quantum mechanical description[Gro46,BM49,Wig32,Les84,CZ99,ZC99].

Naturally, this rigid rotation in phase-space preserves areas, and thus automatically illustrates the uncertainty principle. By contrast, in general, in the conventional formulation of quantum mechanics, this result is deprived of intuitive import, or, at the very least, simplicity: upon integration in $x$ (or $p$) to yield usual marginal probability densities, the rotation induces apparent complicated shape variations of the oscillating probability density profile, such as wavepacket spreading (as evident in the shadow projections on the $x$ and $p$ axes of Fig. 4 ).

Only when (as is the case for coherent states[Sch88,CUZ01,HSD95,Sam00]) a Wigner function configuration has an *additional* axial $x-p$ symmetry around its *own* center, will it possess an invariant profile upon this rotation, and hence a shape-invariant oscillating probability density[ZC99].

In Dirac's interaction representation, a more complicated interaction Hamiltonian superposed on the oscillator one, leads to shape changes of the WF configurations placed on the above "turntable", and serves to generalize to scalar field theory[CZ99].

## 0.8 Non-diagonal Wigner Functions

More generally, to represent all operators on phase-space in a selected basis, one looks at the Weyl-correspondents of arbitrary $|a\rangle\langle b|$, referred to as *non-diagonal WF*s [Gro46]. These enable investigation of interference phenomena and the transition amplitudes in the formulation of quantum mechanical perturbation theory [BM49,WO88,CUZ01].

Both the diagonal and the non-diagonal WFs are represented in (2), by replacing $\boldsymbol{\rho} \to |\psi_a\rangle\langle\psi_b|$,

$$\begin{aligned} f_{ba}(x,p) &\equiv \frac{1}{2\pi}\int dy\; e^{-iyp}\left\langle x+\frac{\hbar}{2}y\middle|\psi_a\right\rangle \left\langle \psi_b\middle| x-\frac{\hbar}{2}y\right\rangle \\ &= \frac{1}{2\pi}\int dy e^{-iyp}\psi_b^*\left(x-\frac{\hbar}{2}y\right)\psi_a\left(x+\frac{\hbar}{2}y\right) = f_{ab}^*(x,p) \\ &= \psi_a(x)\star\delta(p)\star\psi_b^*(x)\ , \end{aligned} \tag{63}$$

The representation on the last line is due to [Bra94] and lends itself to a more compact and elegant proof of Lemma 0.1.

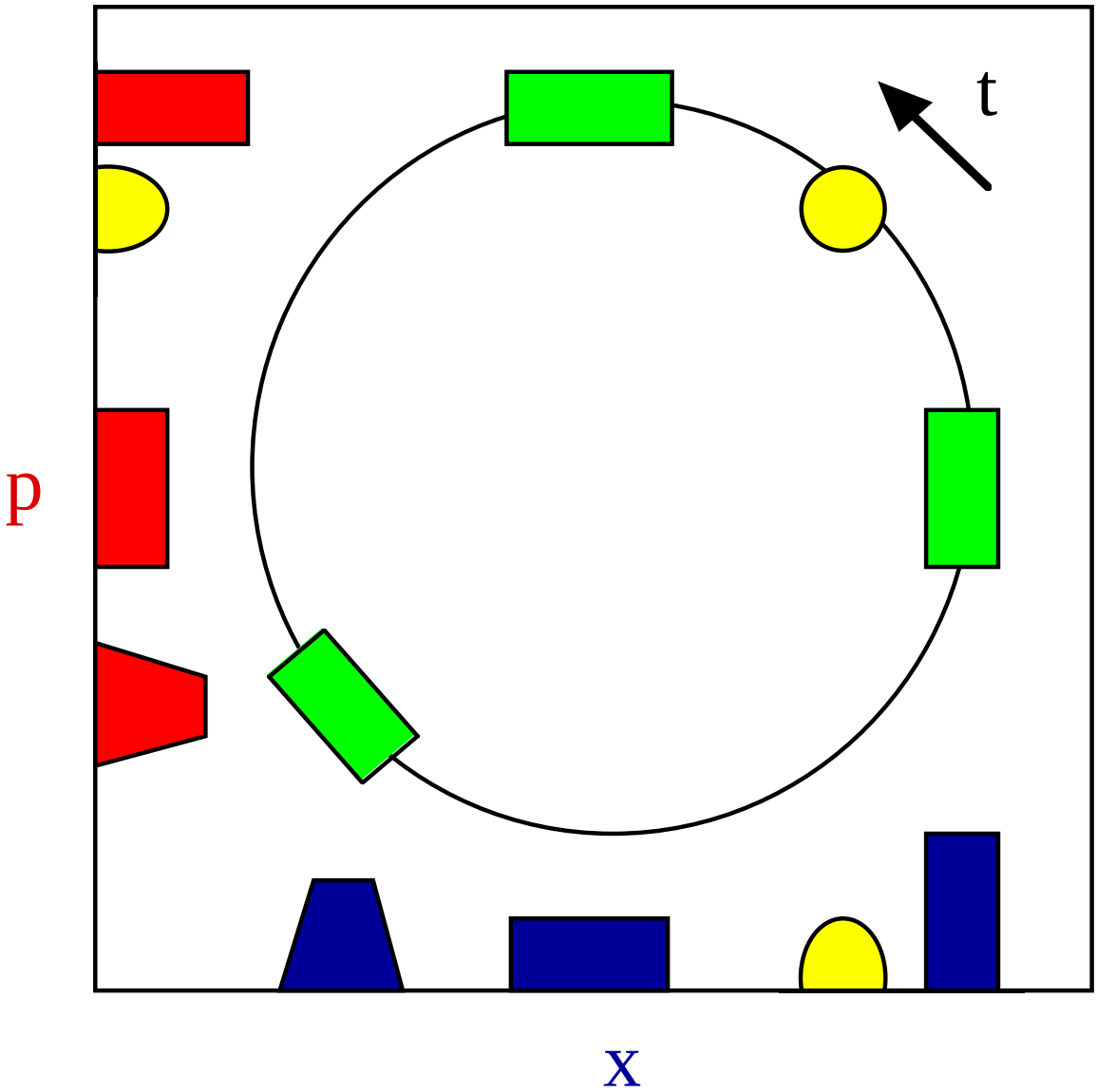


Figure 4. Time evolution of generic WF configurations driven by an oscillator Hamiltonian. The $t$-arrow indicates the rotation sense of $x$ and $p$, and so, for fixed $x$ and $p$ axes, the WF shoebox configurations rotate rigidly in the opposite direction, clockwise. (The sharp angles of the WFs in the illustration are actually unphysical, and were only chosen to monitor their "spreading wavepacket" projections more conspicuously.) These $x$ and $p$-projections (shadows) are meant to be intensity profiles on those axes, but are expanded on the plane to aid visualization. The circular figure represents a coherent state, a Gaussian displaced off the origin, which projects on either axis identically at all times, thus without shape alteration of its wavepacket through time evolution.

Just as pure-state diagonal WFs obey a projection condition, so too the non-diagonals. For wave functions which are orthonormal for discrete state labels, $\int dx\, \psi_a^*(x)\psi_b(x) = \delta_{ab}$, the transition amplitude collapses to

$$\int dxdp\, f_{ab}\,(x,p) = \delta_{ab}\ . \tag{64}$$

To perform spectral operations analogous to those of Hilbert space, it is useful to note that these WFs are $\star$-orthogonal[Fai64]

$$(2\pi\hbar)\ f_{ba} \star f_{dc} = \delta_{bc} f_{da}\ , \tag{65}$$

as well as complete[Moy49] for integrable functions on phase space,

$$(2\pi\hbar) \sum_{a,b} f_{ab}\,(x_1,p_1)\ f_{ba}\,(x_2,p_2) = \delta\,(x_1-x_2)\ \delta\,(p_1-p_2)\ . \tag{66}$$

For example, for the SHO in one dimension, non-diagonal WFs are

$$f_{kn} = \frac{1}{\sqrt{n!k!}}\,(a^*\star)^n\ f_0\ (\star a)^k\,, \qquad f_0 = \frac{1}{\pi\hbar} e^{-\left(x^2+p^2\right)/\hbar}\,, \tag{67}$$

(cf. coherent states [CUZ01,Sch88,DG80]). The $f_{0n}$ are readily identifiable[GLS68], up to a phase-space Gaussian ($f_0$), with the analytic Bargmann representation of wavefunctions: note that

$$(a^*\star)^n\, f_0 = f_0\ (2a^*)^n,$$

mere functions free of operators, where $a^* = a^\dagger$, amounts to Bargmann's variable $z$. (Also note the limit $L_0^n = 1$ below.)

Explicitly, in terms of associated Laguerre polynomials, these are[Gro46,BM49,Fai64]

$$f_{kn} = \sqrt{\frac{k!}{n!}}\ e^{i(k-n)\ \arctan(p/x)}\ \frac{(-1)^k}{\pi\hbar}\left(\frac{x^2+p^2}{\hbar/2}\right)^{(n-k)/2}\ L_k^{n-k}\left(\frac{x^2+p^2}{\hbar/2}\right)\ e^{-\left(x^2+p^2\right)/\hbar}. \tag{68}$$

These SHO non-diagonal WFs are direct solutions to [Fai64]

$$H \star f_{kn} = E_n\, f_{kn}\ , \qquad\qquad f_{kn} \star H = E_k\, f_{kn}\ . \tag{69}$$

The resulting energy $\star$-genvalue conditions are $\left(E_n - \frac{1}{2}\right)/\hbar = n$, an integer; and $\left(E_k - \frac{1}{2}\right)/\hbar = k$, also an integer.

The general spectral theory of WFs is covered in [BFF78,FM91,Lie90,BDW99,CUZ01].

## 0.9 Stationary Perturbation Theory

Given the spectral properties summarized, the phase-space perturbation formalism is self-contained, and it need not make reference to the Hilbert-space treatment [BM49,WO88,CUZ01,SS02,MS96].

For a perturbed Hamiltonian,

$$H\,(x,p) = H_0(x,p) + \lambda\ H_1(x,p)\ , \tag{70}$$

seek a formal series solution,

$$f_n\,(x,p) = \sum_{k=0}^{\infty} \lambda^k f_n^{(k)}(x,p), \qquad\qquad E_n = \sum_{k=0}^{\infty} \lambda^k E_n^{(k)}, \tag{71}$$

of the left-right-$\star$-genvalue equations (17), $H \star f_n = E_n f_n = f_n \star H$.

Matching powers of $\lambda$ in the eigenvalue equation[CUZ01],

$$E_n^{(0)} = \int dxdp\, f_n^{(0)}(x,p)\ H_0(x,p), \qquad\qquad E_n^{(1)} = \int dxdp\, f_n^{(0)}(x,p)\ H_1(x,p), \tag{72}$$

$$\begin{aligned} f_n^{(1)}\,(x,p) &= \sum_{k\neq n} \frac{f_{kn}^{(0)}\,(x,p)}{E_n^{(0)} - E_k^{(0)}} \int dXdP\, f_{nk}^{(0)}\,(X,P)\ \ H_1\,(X,P) \\ &\quad + \sum_{k\neq n} \frac{f_{nk}^{(0)}\,(x,p)}{E_n^{(0)} - E_k^{(0)}} \int dXdP\, f_{kn}^{(0)}\,(X,P)\ H_1\,(X,P)\ . \end{aligned} \tag{73}$$

**Example.** Consider all polynomial perturbations of the harmonic oscillator in a unified treatment, by choosing

$$H_1 = e^{\gamma x + \delta p} = e_\star^{\gamma x + \delta p} = \left(e^{\gamma x} \star e^{\delta p}\right)\ e^{i\gamma\delta/2} = \left(e^{\delta p} \star e^{\gamma x}\right)\ e^{-i\gamma\delta/2}\ , \tag{74}$$

to evaluate a generating function for all the first-order corrections to the energies[CUZ01],

$$E^{(1)}(s) \equiv \sum_{n=0}^{\infty} s^n E_n^{(1)} = \int dxdp \sum_{n=0}^{\infty} s^n f_n^{(0)} H_1\ , \tag{75}$$

hence

$$E_n^{(1)} = \frac{1}{n!}\ \frac{d^n}{ds^n}\ E^{(1)}(s)\Big|_{s=0}\ . \tag{76}$$

From the spectral resolution (56) and the explicit form of the $\star$-exponential of the oscillator Hamiltonian (58) (with $e^{it} \to s$ and $E_n^{(0)} = \left(n + \frac{1}{2}\right)\hbar$), it follows that

$$\sum_{n=0}^{\infty} s^n f_n^{(0)} = \frac{1}{\pi\hbar(1+s)}\ \exp\left(\frac{x^2+p^2}{\hbar}\ \ \frac{s-1}{s+1}\right)\ , \tag{77}$$

and hence

$$\begin{aligned} E^{(1)}\,(s) &= \frac{1}{\pi\hbar\,(1+s)} \int dxdp\, e^{\gamma x + \delta p} \exp\left(-\frac{x^2+p^2}{\hbar}\ \ \frac{1-s}{1+s}\right) \\ &= \frac{1}{1-s} \exp\left(\frac{\hbar}{4}\left(\gamma^2+\delta^2\right)\frac{1+s}{1-s}\right)\ . \end{aligned} \tag{78}$$

E.g., specifically,

$$\begin{aligned} E_0^{(1)} &= \exp\left(\frac{\hbar}{4}\left(\gamma^2+\delta^2\right)\right)\ , \qquad\qquad E_1^{(1)} = \left(1 + \frac{\hbar}{2}\left(\gamma^2+\delta^2\right)\right)\ E_0^{(1)}\ , \\ E_2^{(1)} &= \left(1 + \hbar\left(\gamma^2+\delta^2\right) + \frac{\hbar^2}{8}\left(\gamma^2+\delta^2\right)^2\right)\ E_0^{(1)}\ , \end{aligned} \tag{79}$$

and so on. All the first order corrections to the energies are even functions of the parameters: only even functions of $x$ and $p$ can contribute to first-order shifts in the oscillator energies.

First-order corrections to the WFs may be similarly calculated using generating functions for non-diagonal WFs. Higher order corrections are straightforward but tedious. Degenerate perturbation theory also admits an autonomous formulation in phase-space, equivalent to Hilbert space and path-integral treatments.

### 0.10 Propagators

Time evolution of general WFs beyond the above treatment is discussed at length in refs [BM49,Tak54,Ber75,GM80,CUZ01,BR93,Wo82,Wo02,FM03,TW03].

A further application of the spectral techniques outlined is the computation of the WF time-evolution operator from the propagator for wave functions, which is given as a bilinear sum of energy eigenfunctions,

$$G(x,X;t) = \sum_a \psi_a(x)\, e^{-iE_a t/\hbar}\, \psi_a^*(X) \equiv \exp\Big(iA_{eff}(x,X;t)\Big) \ , \tag{80}$$

as it may be thought of as an exponentiated effective action. (Henceforth in this section, take $\hbar = 1$).

This leads directly to a similar bilinear double sum for the WF time-transformation kernel [Moy49],

$$T(x,p;X,P;t) = 2\pi \sum_{a,b} f_{ba}(x,p)\, e^{-i(E_a-E_b)t}\, f_{ab}(X,P) \ . \tag{81}$$

Defining a "big star" operation as a $\star$-product for the upper-case (initial) phase-space variables,

$$\bigstar \equiv e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_X \overrightarrow{\partial}_P - \overleftarrow{\partial}_P \overrightarrow{\partial}_X)} \ , \tag{82}$$

it follows that

$$T(x,p;X,P;t)\bigstar f_{dc}(X,P) = \sum_b f_{bc}(x,p)\, e^{-i(E_c-E_b)t}\, f_{db}(X,P) \ , \tag{83}$$

hence, cf. (55),

$$\int dX dP\, T(x,p;X,P;t) f_{dc}(X,P) = f_{dc}(x,p) e^{-i(E_c-E_d)t} = U_\star^{-1} \star f_{dc}(x,p;0) \star U_\star = f_{dc}(x,p;t). \tag{84}$$

**Example.** For a free particle of unit mass in one dimension, $H = p^2/2$, WFs propagate according to

$$\begin{aligned} & T_{free}\,(x,p;X,P;t) \\ &= \frac{1}{2\pi} \int dk \int dq\, e^{i(k-q)x}\, \delta\left(p - \frac{1}{2}(k+q)\right)\, e^{-i\left(q^2-k^2\right)t/2}\, e^{-i(k-q)X}\, \delta\left(P - \frac{1}{2}(k+q)\right) \\ &= \delta\,(x - X - Pt)\, \delta\,(p - P) \ , \end{aligned} \tag{85}$$

amounting to "classical" free motion,

$$f(x,p;t) = f(x-pt,p;0)\ . \tag{86}$$

### 0.11 Canonical Transformations

Canonical transformations $(x,p) \mapsto (X(x,p),P(x,p))$ preserve the phase-space volume (area) element (again, take $\hbar = 1$) through a trivial Jacobian,

$$dXdP = dxdp\ \{X,P\}\ , \tag{87}$$

i.e., they preserve Poisson Brackets

$$\{u,v\}_{xp} \equiv \frac{\partial u}{\partial x}\frac{\partial v}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial v}{\partial x}\ , \tag{88}$$

$$\{X,P\}_{xp} = 1, \qquad \{x,p\}_{XP} = 1. \tag{89}$$

Upon quantization, the c-number function Hamiltonian transforms "classically", $\mathcal{H}(X,P) \equiv H(x,p)$, like a scalar. Does the $\star$-product remain invariant under this transformation?

Yes, for *linear* canonical transformations[HKN88], but clearly *not for general canonical transformations*. Still, things can be put right, by devising general *covariant* transformation rules for the $\star$-product[CFZ98]: The WF transforms in comportance with Dirac's quantum canonical transformation theory[Dir33].

In conventional quantum mechanics, for classical canonical transformations generated by $F_{cl}(x,X)$,

$$p = \frac{\partial F_{cl}(x,X)}{\partial x}\ , \qquad P = -\frac{\partial F_{cl}(x,X)}{\partial X}\ , \tag{90}$$

the energy eigenfunctions transform in a generalization of the "representation-changing" Fourier transform[Dir33],

$$\psi_E(x) = N_E \int dX\, e^{iF(x,X)}\, \Psi_E(X)\ . \tag{91}$$

(In this expression, the generating function $F$ may contain $\hbar$ corrections[BCT82] to the classical one, in general—but for several simple quantum mechanical systems it manages not to[CG92,DG02].) Hence[CFZ98], there is a transformation functional for WFs, $\mathcal{T}(x,p;X,P)$, such that

$$f(x,p) = \int dXdP\ \mathcal{T}(x,p;X,P) \bigstar \mathcal{F}(X,P) = \int dXdP\ \mathcal{T}(x,p;X,P)\ \mathcal{F}(X,P)\ , \tag{92}$$

where

$$\begin{aligned} &\mathcal{T}\ (x,p;X,P) \\ &= \frac{|N|^2}{2\pi}\int dYdy\ \exp\left(-iyp + iPY - iF^*(x-\frac{y}{2},X-\frac{Y}{2}) + iF(x+\frac{y}{2},X+\frac{Y}{2})\right). \end{aligned} \tag{93}$$

Moreover, it can be shown that[CFZ98],

$$H(x,p) \star \mathcal{T}(x,p;X,P) = \mathcal{T}(x,p;X,P) \bigstar \mathcal{H}(X,P). \tag{94}$$

That is, if $\mathcal{F}$ satisfies a $\bigstar-$genvalue equation, then $f$ satisfies a $\star$-genvalue equation with the same eigenvalue, and vice versa. This proves useful in constructing WFs for simple systems which can be trivialized classically through canonical transformations.

A thorough discussion of MB automorphisms may start from ref [BCW02]. (Also see [Hie82,DKM88,GR94,DV97,Hak99,KL99,DP01].)

Time evolution is a canonical transformation [Dir33], with the generator's role played by the effective action $A$ of the previous section, incorporating quantum corrections to both phases and normalizations; it connects initial wave functions to those at a final time.

**Example.** For the linear potential, with

$$H = p^2 + x\ , \tag{95}$$

wave function evolution is determined by the propagator

$$\exp\left(iA_{lin}(x,X;t)\right) = \frac{1}{\sqrt{4\pi i t}} \exp\left(\frac{i\,(x-X)^2}{4t} - \frac{i\,(x+X)\,t}{2} - \frac{it^3}{12}\right). \tag{96}$$

$T$ then evaluates to

$$\begin{aligned}
&T_{lin}\,(x,p;X,P;t)\\
&= \frac{1}{2\pi}\int dY dy\, \exp\left(-iyp + iPY - iA^*_{lin}(x-\frac{y}{2}, X-\frac{Y}{2};t) + iA_{lin}(x+\frac{y}{2}, X+\frac{Y}{2};t)\right)\\
&= \frac{1}{8\pi^2 t}\int dY dy\, \exp\left(-iyp + iPY - \frac{it}{2}\,(y+Y) + \frac{i}{2t}(x-X)(y-Y)\right)\\
&= \frac{1}{2t}\,\delta\left(p + \frac{t}{2} - \frac{x-X}{2t}\right)\,\delta\left(P - \frac{t}{2} - \frac{x-X}{2t}\right)\\
&= \delta\left(p+t-P\right)\,\delta\left(x - 2tp - t^2 - X\right)\\
&= \delta\left(x - X - (p+P)\,t\right)\,\delta\left(P - p - t\right).
\end{aligned} \tag{97}$$

The $\delta$-functions enforce exactly the classical motion for a mass$=1/2$ particle subject to a negative constant force of unit magnitude (acceleration $=-2$). Thus the WF evolves "classically" as

$$f(x,p;t) = f(x - 2pt - t^2, p + t\ ;0). \tag{98}$$

NB. Time-*in*dependence follows for $f(x,p;0)$ being any function of the energy variable, since $x + p^2 = x - 2pt - t^2 + (p+t)^2$.

The evolution kernel $T$ propagates an arbitrary WF through just

$$f(x,p;t) = \int dX dP\ T(x,p;X,P;t)\ f(X,P;0)\ . \tag{99}$$

The underlying phase-space structure, however, is more evident if one of the wave-function propagators is given in coordinate space, and the other in momentum space. Then the

path integral expressions for the two propagators can be combined into a single phase-space path integral. For every time increment, phase space is integrated over to produce the new Wigner function from its immediate ancestor. The result is

$$
\begin{aligned}
&T\ (x,p;X,P;t) \qquad (100)\\
&=\frac{1}{\pi^2}\int dx_1dp_1\int dx_2dp_2 e^{2i(x-x_1)(p-p_1)}e^{-ix_1p_1}\ \langle x_1;t|x_2;0\rangle\ \langle p_1;t|p_2;0\rangle^*\ e^{ix_2p_2}e^{-2i(X-x_2)(P-p_2)},
\end{aligned}
$$

where $\langle x_1;t\ |x_2;0\rangle$ and $\langle p_1;t\ |p_2;0\rangle$ are the path integral expressions in coordinate space, and in momentum space. Blending these $x$ and $p$ path integrals gives a genuine path integral over phase space [Ber80,DK85]. For a direct connection of $U_\star$ to this integral, see ref[Sha79,Lea68,Sam00].

## 0.12 The Weyl Correspondence

This section summarizes the bridge and equivalence of phase-space quantization to the conventional operator formulation of quantum mechanics in Hilbert space. The Weyl correspondence merely provides a change of representation between phase space and Hilbert space. In itself, it does not map (commutative) classical mechanics to (non-commutative) quantum mechanics ("quantization"), as Weyl had originally hoped. But it makes the deformation map at the heart of quantization easier to grasp, now defined within a common representation, and thus more intuitive.

Weyl[Wey27] introduced an association rule mapping invertibly c-number phase-space functions $g(x,p)$ (called phase-space kernels) to operators $\mathfrak{G}$ in a given ordering prescription. Specifically, $p\mapsto\mathfrak{p}$, $x\mapsto\mathfrak{x}$, and, in general,

$$
\mathfrak{G}(\mathfrak{x},\mathfrak{p})=\frac{1}{(2\pi)^2}\int d\tau d\sigma dxdp\ g(x,p)\exp\Big(i\tau(\mathfrak{p}-p)+i\sigma(\mathfrak{x}-x)\Big). \qquad (101)
$$

The eponymous ordering prescription requires that an arbitrary operator, regarded as a power series in $\mathfrak{x}$ and $\mathfrak{p}$, be first ordered in a completely symmetrized expression in $\mathfrak{x}$ and $\mathfrak{p}$, by use of Heisenberg's commutation relations, $[\mathfrak{x},\mathfrak{p}]=i\hbar$.

A term with $m$ powers of $\mathfrak{p}$ and $n$ powers of $\mathfrak{x}$ is obtained from the coefficient of $\tau^m\sigma^n$ in the expansion of $(\tau\mathfrak{p}+\sigma\mathfrak{x})^{m+n}$, which serves as a generating function of Weyl-ordered polynomials[GF91]. It is evident how the map yields a Weyl-ordered operator from a polynomial phase-space kernel. It includes every possible ordering with multiplicity one, e.g.,

$$
6p^2x^2 \quad\mapsto\quad \mathfrak{p}^2\mathfrak{x}^2+\mathfrak{x}^2\mathfrak{p}^2+\mathfrak{p}\mathfrak{x}\mathfrak{p}\mathfrak{x}+\mathfrak{p}\mathfrak{x}^2\mathfrak{p}+\mathfrak{x}\mathfrak{p}\mathfrak{x}\mathfrak{p}+\mathfrak{x}\mathfrak{p}^2\mathfrak{x}\ . \qquad (102)
$$

In general[McC32],

$$
p^mx^n \quad\mapsto\quad \frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r}\mathfrak{x}^r\mathfrak{p}^m\mathfrak{x}^{n-r}=\frac{1}{2^m}\sum_{s=0}^{m}\binom{m}{s}\mathfrak{p}^s\mathfrak{x}^n\mathfrak{p}^{m-s}. \qquad (103)
$$

Phase-space constants map to the constant multiplying $\mathbb{1}$, the identity in Hilbert space.

In this correspondence scheme, then,

$$h\ \mathrm{Tr}\mathfrak{G} = \int dxdp\ g\ . \tag{104}$$

Conversely[Dir30,Gro46,Kub64,Lea68,HOS84], the c-number phase-space kernels $g(x,p)$ of Weyl-ordered operators $\mathfrak{G}(\mathfrak{x},\mathfrak{p})$ are specified by $\mathfrak{p} \mapsto p$, $\mathfrak{x} \mapsto x$, or, more precisely, by the "Wigner map",

$$\begin{aligned} g(x,p) &= \frac{\hbar}{2\pi}\int d\tau d\sigma\ e^{i(\tau p+\sigma x)}\mathrm{Tr}\ \left(e^{-i(\tau\mathfrak{p}+\sigma\mathfrak{x})}\mathfrak{G}\right) \\ &= \hbar\int dy\ e^{-iyp}\left\langle x+\frac{\hbar}{2}y\right|\mathfrak{G}(\mathfrak{x},\mathfrak{p})\left|x-\frac{\hbar}{2}y\right\rangle, \end{aligned} \tag{105}$$

since the above trace reduces to

$$\int dz\ e^{i\tau\sigma\hbar/2}\langle z|e^{-i\sigma\mathfrak{x}}e^{-i\tau\mathfrak{p}}\mathfrak{G}|z\rangle = \int dz e^{i\sigma(\tau\hbar/2-z)}\langle z-\hbar\tau|\mathfrak{G}|z\rangle. \tag{106}$$

Equivalently,

$$\langle x|\mathfrak{G}|y\rangle = \int\frac{dp}{2\pi\hbar}\ \exp\left(ip\frac{(x-y)}{\hbar}\right)\ g\left(\frac{x+y}{2},p\right).$$

Thus, the density matrix $|\psi\rangle\langle\psi|/h$ inserted in this expression[Moy49] yields the hermitean generalization of the Wigner function (63) encountered,

$$\begin{aligned} f_{ab}(x,p) &\equiv \frac{1}{2\pi}\int dy\ e^{-iyp}\left\langle x+\frac{\hbar}{2}y\ \middle|\ \psi_b\right\rangle\ \left\langle\ \psi_a\ \middle|x-\frac{\hbar}{2}y\right\rangle \\ &= \frac{1}{2\pi}\int dye^{-iyp}\psi_a^*\left(x-\frac{\hbar}{2}y\right)\ \psi_b\left(x+\frac{\hbar}{2}y\right) = \ f_{ba}^*(x,p)\ , \end{aligned} \tag{107}$$

where the $\psi_a(x)$s are (ortho-)normalized solutions of a Schrödinger problem. (Wigner[Wig32] mainly considered the diagonal elements of the pure-state density matrix, denoted above as $f_m \equiv f_{mm}$.) As a consequence, matrix elements of operators, i.e., traces of them with the density matrix, are produced through mere phase-space integrals[Moy49],

$$\langle\psi_m|\mathfrak{G}|\psi_n\rangle = \int dxdp\ g(x,p)f_{mn}(x,p), \tag{108}$$

and thus expectation values follow for $m = n$, as utilized throughout in this overview. Hence,

$$\langle\psi_m|\exp i(\sigma\mathfrak{x}+\tau\mathfrak{p})|\psi_m\rangle = \int dxdp\ f_m(x,p)\exp i(\sigma x+\tau p), \tag{109}$$

the celebrated moment-generating functional[Moy49] of the Wigner distribution.

Products of Weyl-ordered operators are not necessarily Weyl-ordered, but may be easily reordered into unique Weyl-ordered operators through the degenerate Campbell-Baker-Hausdorff identity. In a study of the uniqueness of the Schrödinger representation, von

Neumann[Neu31] adumbrated the composition rule of kernel functions in such operator products, appreciating that Weyl's correspondence was in fact a homomorphism. (Effectively, he arrived at the Fourier space convolution representation of the star product below.)

Finally, Groenewold[Gro46] neatly worked out in detail how the kernel functions $f$ and $g$ of two operators $\mathfrak{F}$ and $\mathfrak{G}$ must compose to yield the kernel (the Wigner map image, sometimes called the "Weyl symbol") of $\mathfrak{F}\ \mathfrak{G}$,

$$\mathfrak{F}\ \mathfrak{G} = \tfrac{1}{(2\pi)^4} \int d\xi d\eta d\xi' d\eta' dx' dx'' dp' dp'' f(x',p') g(x'',p'') \times \exp i(\xi(\mathfrak{p}-p') + \eta(\mathfrak{x}-x'))\ \exp i(\xi'(\mathfrak{p}-p'') + \eta'(\mathfrak{x}-x'')) =$$

$$= \frac{1}{(2\pi)^4} \int d\xi d\eta d\xi' d\eta' dx' dx'' dp' dp'' f(x',p') g(x'',p'') \exp i\Big((\xi+\xi')\mathfrak{p} + (\eta+\eta')\mathfrak{x}\Big)$$

$$\times\ \exp i\left(-\xi p' - \eta x' - \xi' p'' - \eta' x'' + \frac{\hbar}{2}(\xi\eta' - \eta\xi')\right). \tag{110}$$

Changing integration variables to

$$\xi' \equiv \frac{2}{\hbar}(x-x'), \quad \xi \equiv \tau - \frac{2}{\hbar}(x-x'), \quad \eta' \equiv \frac{2}{\hbar}(p'-p), \quad \eta \equiv \sigma - \frac{2}{\hbar}(p'-p), \tag{111}$$

reduces the above integral to the fundamental isomorphism,

**Theorem 0.1**

$$\mathfrak{F}\ \mathfrak{G} = \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp\ \exp i\Big(\tau(\mathfrak{p}-p) + \sigma(\mathfrak{x}-x)\Big)\ (f \star g)(x,p), \tag{112}$$

*where $f \star g$ is the expression (13).*

□

Noncommutative operator multiplication Wigner-transforms to $\star$-multiplication.

The $\star$-product thus defines the transition from classical to quantum mechanics. In fact, the failure of Weyl-ordered operators to close under multiplication may be stood on its head [Bra02], to *define* a Weyl-symmetrizing operator product, which is commutative and constitutes the Weyl transform of $fg$ instead of the noncommutative $f \star g$. (For example,

$$2x \star p = 2xp + i\hbar \quad \mapsto \quad 2\mathfrak{x}\mathfrak{p} = \mathfrak{x}\mathfrak{p} + \mathfrak{p}\mathfrak{x} + i\hbar.$$

The classical piece of $2x \star p$ maps to the Weyl-symmetrization of the operator product, $2xp \mapsto \mathfrak{x}\mathfrak{p} + \mathfrak{p}\mathfrak{x}$.) One may then solve for the PB in terms of the MB, and, through the Weyl correspondence, reformulate Classical Mechanics in Hilbert space as a deformation of Quantum Mechanics, instead of the other way around [Bra02].

Arbitrary operators $\mathfrak{G}(\mathfrak{x},\mathfrak{p})$ consisting of operators $\mathfrak{x}$ and $\mathfrak{p}$, in various orderings, but with the same classical limit, could be imagined rearranged by use of Heisenberg commutations to canonical completely symmetrized Weyl-ordered forms, in general with $O(\hbar)$ terms generated in the process. Each one might then be inverse-transformed uniquely to its Weyl-correspondent c-number kernel function $g$ in phase space. (In practice[Kub64], there is the

above more direct Wigner transform formula (105), which bypasses any need for an actual rearrangement.) Thus, operators differing from each other by different orderings of their $\mathfrak{x}$s and $\mathfrak{p}$s Wigner-map to kernel functions $g$ coinciding with each other at $O(\hbar^0)$, but different at $O(\hbar)$, in general.

Hence, in phase-space quantization, a survey of all alternate operator orderings in a problem with such ambiguities amounts to a survey of the "quantum correction" $O(\hbar)$ pieces of the respective kernel functions, i.e. the Wigner transforms of those operators, and their study is systematized and expedited. Choice-of-ordering problems then reduce to purely $\star$-product algebraic ones, as the resulting preferred orderings are specified through particular deformations in the c-number kernel expressions resulting from the particular solution in phase space[CZ02].

### 0.13 Alternate Rules of Association

The Weyl correspondence rule (101) is not unique: there are a host of alternate *equivalent* association rules which specify corresponding representations. All these representations with equivalent formalisms are typified by characteristic quasi-distribution functions and $\star$-products, all inter-convertible among themselves. They have been surveyed comparatively and organized in [Lee95,BJ84], on the basis of seminal classification work by Cohen [Coh66,Coh76], and are favored by virtue of their different characteristic properties in varying applications.

For example, instead of the symmetric operator $\exp(i\tau\mathfrak{p} + i\sigma\mathfrak{x})$ underlying the Weyl transform, one might posit, instead [Lee95,HOS84], antistandard ordering,

$$\exp(i\tau\mathfrak{p})\exp(i\sigma\mathfrak{x}) = \exp(i\tau\mathfrak{p} + i\sigma\mathfrak{x})w(\tau,\sigma), \tag{113}$$

with $w = \exp(i\hbar\tau\sigma/2)$, which specifies the Kirkwood-Rihaczek prescription[Kir33]; or else standard ordering (momenta to the right), $w = \exp(-i\hbar\tau\sigma/2)$ on the right-hand-side of the above, for the Mehta prescription, also utilized by Moyal[Moy49]; or their (real) average, $w = \cos(\hbar\tau\sigma/2)$ for the (older) Rivier prescription [Ter37]; or normal and antinormal orderings for the Glauber-Sudarshan prescriptions, generalizing to $w = \exp(\frac{\hbar}{4}(\tau^2+\sigma^2))$ for the Husimi prescription [Hus40,Tak89]; or $w = \sin(\hbar\tau\sigma/2)/(\hbar\tau\sigma/2)$, for the Born-Jordan prescription; and so on.

The corresponding quasi-distribution functions in each representation can be obtained systematically as convolution transforms of each other[Coh76,Lee95,HOS84]; and, likewise, the kernel function observables are convolution "dressings" of each other, as are their $\star$-products [Dun88,AW70,Ber75].

**Example.** For instance, the Husimi distribution follows from a "Gaussian smoothing" invertible linear conversion map [WO87,Tak89,Lee95] of the WF,

$$\begin{aligned} f_{_H} = T(f) &= \exp\left(\frac{\hbar}{4}(\partial_x^2 + \partial_p^2)\right) f \\ &= \frac{1}{\pi\hbar}\int dx'dp' \exp\left(-\frac{(x'-x)^2 + (p'-p)^2)}{\hbar}\right) f(x',p'), \end{aligned} \tag{114}$$

and likewise for the observables, the *same operators* $\mathfrak{G}$ in this alternate ordering, so that

$$\langle \mathfrak{G} \rangle = \int dxdp\ g(x,p)\ \exp\left(-\frac{\hbar}{4}(\partial_x^2+\partial_p^2)\right)\ f_H = \int dxdp\ g_H\ e^{\hbar(\overleftarrow{\partial}_x\overrightarrow{\partial}_x+\overleftarrow{\partial}_p\overrightarrow{\partial}_p)/2}\ f_H. \quad (115)$$

That is, expectation values of observables now entail equivalence conversion dressings of the respective kernel functions and a corresponding isomorph $\star$-product [Ba79,OW81,Vor89,Tak89,Zac00],

$$\circledast = \exp\left(\frac{\hbar}{2}(\overleftarrow{\partial}_x\overrightarrow{\partial}_x+\overleftarrow{\partial}_p\overrightarrow{\partial}_p)\right) \quad \star \ = \exp\left(\frac{\hbar}{2}(\overleftarrow{\partial}_x -i\ \overleftarrow{\partial}_p)(\overrightarrow{\partial}_x +i\ \overrightarrow{\partial}_p)\right),$$

cf. (120) below. Evidently, however, this one now *cannot be simply dropped inside integrals*, quite unlike the case of the WF (16).

For this reason, distributions such as this Husimi distribution (which is actually positive semi-definite [deB67,Car76,OW81,Ste80]) *cannot* be automatically thought of as bona-fide probability distributions.

This is often dramatized as the failure of the Husimi distribution $f_H$ to yield the correct $x$- or $p$-marginal probabilities, upon integration by $p$ or $x$, respectively[OW81,HOS84]. Since phase-space integrals are thus complicated by conversion dressing convolutions, they preclude direct applications of the Schwarz inequality and the standard inequality-based moment-constraining techniques of probability theory, as well as routine completeness and orthonormality based functional analytic operations. (Ignoring the above equivalence dressings and, instead, simply treating the Husimi distribution as an ordinary probability distribution in evaluating expectation values, nevertheless, results in loss of quantum information—effectively "coarse-graining" to a classical limit, and thereby increasing the entropy[Bra94].)

Similar caveats also apply to more recent symplectic tomographic representations [MMT96,MMM01,Leo97], which are also positive semi-definite, but also do not constitute conventional probability distributions.

**Exercise.** One may work out Moyal's inter-relations[Moy49,Coh66,Coh76] between the Weyl-ordering kernel functions and the standard-ordering correspondents; as well as the respective dressing relations between the proper $\star$-products[Lee95], in analogy to the foregoing example for the Husimi prescription. The $w = \exp(-i\hbar\tau\sigma/2)$ mentioned dictates a dressing of kernels, $g_s = T(g) \equiv \exp(-i\hbar\partial_x\partial_p/2)\ g(x,p)$, and of $\star$-products by (120) below. Abstracting the Weyl map functional of Section (0.12), for generic Hilbert-space variables $\mathfrak{z}$ and phase-space variables $z$, the Weyl map compacts to an integral functional kernel[Kub64], $\mathfrak{G}(\mathfrak{z}) = \int dz\Delta(\mathfrak{z},z)g(z)$, and the inverse (Wigner) map to $g(z) = h\mathrm{Tr}(\overline{\Delta}(\mathfrak{z},z)\mathfrak{G}(\mathfrak{z}))$, where $h\mathrm{Tr}\Delta(\mathfrak{z},z)\overline{\Delta}(\mathfrak{z},z') = \delta^2(z-z')$, $\int dz\Delta(\mathfrak{z},z) = \int dz\overline{\Delta}(\mathfrak{z},z) = \mathbb{1}$, and $h\mathrm{Tr}\Delta = h\mathrm{Tr}\overline{\Delta} = 1$. The $\star$-product is thus a convolution in the integral representation, $f\star g = \int dz'dz''f(z')g(z'')(h\ \mathrm{Tr}\overline{\Delta}(\mathfrak{z},z)\Delta(\mathfrak{z},z')\Delta(\mathfrak{z},z''))$. The dressing of these functionals is then $\Delta_s(\mathfrak{z},z) = T^{-1}(z)\Delta(\mathfrak{z},z)$, so that both prescriptions yield the same operator $\mathfrak{G}$, when $g_s(z) = T(z)g(z)$ and $\overline{\Delta}_s = T\overline{\Delta}$. More abstractly then, the integral kernel for $\circledast$ amounts to $h\mathrm{Tr}(T(z)\overline{\Delta}(\mathfrak{z},z)T^{-1}(z')\Delta(\mathfrak{z},z')T^{-1}(z'')\Delta(\mathfrak{z},z''))$.

## 0.14 The Groenewold-van Hove Theorem and the Uniqueness of MBs and $\star$-products

Groenewold's correspondence principle theorem[Gro46] (to which van Hove's extension to all association rules is often attached[vH51]) points out that, in general, there is *no* invertible linear map from *all functions* of phase space $f(x,p), g(x,p), ...,$ to hermitean operators in Hilbert space $\mathfrak{Q}(f)$, $\mathfrak{Q}(g), ...,$ such that the PB structure is preserved,

$$\mathfrak{Q}(\{f,g\}) = \frac{1}{i\hbar}\left[\ \mathfrak{Q}(f), \mathfrak{Q}(g)\ \right], \tag{116}$$

as utilized in Dirac's heuristics.

Instead, the Weyl correspondence map (101) from functions to ordered operators,

$$\mathfrak{W}(f) \equiv \frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp\ f(x,p)\exp(i\tau(\mathfrak{p}-p)+i\sigma(\mathfrak{x}-x)), \tag{117}$$

specifies the $\star$-product in (112), $\mathfrak{W}(f\star g) = \mathfrak{W}(f)\ \mathfrak{W}(g)$, and thus the MB Lie algebra,

$$\mathfrak{W}(\{\!\{f,g\}\!\}) = \frac{1}{i\hbar}\left[\mathfrak{W}(f), \mathfrak{W}(g)\right]. \tag{118}$$

It is the MB, then, instead of the PB, which maps invertibly to the quantum commutator. That is to say, the "deformation" involved in phase-space quantization is nontrivial: the quantum (observable) functions, in general, need not coincide with the classical ones[Gro46], and often involve $O(\hbar)$ corrections, as extensively illustrated in, e.g., refs [CZ02,DS02,CH86]; also see [Got99]. For example, as was already pointed out, the Wigner transform of the square of the angular momentum $\mathfrak{L}\cdot\mathfrak{L}$ turns out to be $L^2 - 3\hbar^2/2$, significantly for the ground-state Bohr orbit [She59,DS82,DS02].

Groenewold's celebrated *counterexample* noted that the vanishing PB expression

$$\{x^3, p^3\} + \frac{1}{12}\{\{p^2, x^3\}, \{x^2, p^3\}\} = 0$$

is anomalous upon implementation of Dirac's proposal to substitute commutators of $\mathfrak{Q}(x), \mathfrak{Q}(p), ...,$ for PBs. Indeed, that substitution, or equivalent substitution of MBs for PBs, yields a *Groenewold anomaly*, $-3\hbar^2$, for this specific expression.

An alternate abstract operator realization of the above MB Lie algebra in phase space (as opposed to the Hilbert space one, $\mathfrak{W}(f)$) linearly is[FFZ89,CFZm98]

$$\mathfrak{K}(f) = f\star\ . \tag{119}$$

Realized on a toroidal phase space, upon a formal identification $\hbar \mapsto 2\pi/N$, this realization of the MB Lie algebra leads to the Lie algebra of $SU(N)$ [FFZ89], by means of Sylvester's clock-and-shift matrices[Syl82]. For generic $\hbar$, it may be thought of as a generalization of $SU(N)$ for continuous $N$. This allows for taking the limit $N \to \infty$ to thus contract to the PB algebra.

Essentially (up to isomorphism), the MB algebra is the unique (Lie) one-parameter deformation (expansion) of the Poisson Bracket algebra[Vey75,BFF78,FLS76,Ar83 Fle90,deW83,BCG97,TD97], a uniqueness extending to the (associative) star product. Isomorphism allows for dressing transformations of the variables (kernel functions and WFs, as in section 0.13 on alternate orderings), through linear maps $f \mapsto T(f)$, which leads to cohomologically equivalent star-product variants, i.e.[Ba79,Vor89,BFF78],

$$T(f \star g) = T(f) \circledast T(g). \tag{120}$$

Consequently, the $\star$-MB algebra is isomorphic to the algebra of $\circledast$-MB.

Computational features of $\star$-products are discussed in refs [BFF78,Han84,RO92,Zac00,EGV89,Vo78,An97,Bra94].

## 0.15 Omitted Miscellany

Phase-space quantization extends in several interesting directions which are not covered in such a summary introduction.

The systematic generalization of the $\star$-product to arbitrary non-flat Poisson manifolds [Kon97], is a culmination of extensions to general symplectic and Kähler geometries [Fed94,Kis01], and varied symplectic contexts [Ber75,RT00,CPP02,BGL01].

For further work on curved spaces, cf. [APW02,BF81,PT99]. For extensive reviews of mathematical issues, cf. [Fol89,Hor79,Wo98,AW70]. For a connection to the theory of modular forms, see [Raj02].

For WFs on discrete phase spaces (finite-state systems), see, among others, [Woo87,KP94,OBB95,ACW98,RA99,RG00,BHP02,MPS02].

Spin is treated in ref [Str57,VG89,AW00]; and forays into a relativistic formulation in [LSU02] (also see ref[CS75,Ran66]).

Inclusion of Electromagnetic fields and gauge invariance is treated in refs [Kub64,Mue99,LF94,LF01,JVS87,ZC99,KO00]. Subtleties of Berry's phase in phase space are addressed in ref [Sam00].

**Selected Papers**

## 0.16 Brief Historical Outline

The decisive contributors to the development of the formulation are Hermann Weyl (1885-1955), Eugene Wigner (1902-1995), Hilbrand Groenewold (1910-1996), and Jose Moyal (1910-1998). The bulk of the theory is implicit in Groenewold's and Moyal's seminal papers.

But this has been a slow story of emerging connections and chains of ever sharper reformulations. Confidence in the autonomy of the formulation accreted slowly. As a result, attribution of critical milestones cannot avoid subjectivity: it cannot automatically highlight merely the *earliest* occurrence of a construct, unless that has also been conclusive enough to yield an "indefinite stay against confusion" about the logical structure of the formulation.

H Weyl (1927)$^{Wey27}$ introduces the correspondence of "Weyl-ordered" operators to phase-space (c-number) kernel functions. The correspondence is based on Weyl's formulation of the Heisenberg group, appreciated through a discrete QM application of Sylvester's (1883)$^{Syl82}$ clock and shift matrices. The correspondence is proposed as a general quantization prescription, unsuccessfully, since it fails, e.g., with angular momentum.

J von Neumann (1931)$^{Neu31}$, in a technical aside off a study of the uniqueness of Schrödinger's representation, based on Weyl's above Heisenberg group formulation, it includes a Fourier transform version of the $\star$-product. This then promotes Weyl's correspondence rule to full isomorphism between Weyl-ordered operator multiplication and $\star$-convolution of kernel functions. Nevertheless, this result is not properly appreciated in von Neumann's celebrated own book on the Foundations of QM.

E Wigner (1932)$^{Wig32}$ introduces the eponymous phase-space distribution function controlling quantum mechanical diffusive flow in phase space. It specifies the time evolution of this function and applies it to quantum statistical mechanics. (Actually, Dirac (1930)$^{Dir30}$ has already considered a formally identical construct for the electron density in a multi-electron Thomas-Fermi atom, but interprets negative values as a failure of the semiclassical approximation, and dismisses the full quantum object.)

H Groenewold (1946)$^{Gro46}$, a seminal but somewhat unappreciated paper, is based on Groenewold's thesis work. It achieves full understanding of the Weyl correspondence as an invertible transform, rather than as a consistent quantization rule. It produces the WF as the phase-space (Weyl) kernel of the density matrix. It reinvents and streamlines von Neumann's construct into the standard $\star$-product, in a systematic exploration of the isomorphism between Weyl-ordered operator products and their kernel function compositions. It thus demonstrates how Poisson Brackets contrast crucially to quantum commutators—"Groenewold's Theorem". By way of illustration, it further works out the harmonic oscillator WF.

J Moyal (1949)$^{Moy49}$ amounts to a grand synthesis: It establishes an independent formulation of quantum mechanics in phase space. It systematically studies all expectation values

of Weyl-ordered operators, and identifies the Fourier transform of their moment-generating function (their characteristic function) to the Wigner Function. It further interprets the subtlety of the "negative probability" formalism and reconciles it with the uncertainty principle and the diffusion of the probability fluid. Not least, it recasts the time evolution of the Wigner Function through a deformation of the Poisson Bracket into the Moyal Bracket (the commutator of $\star$-products, i.e., the Wigner transform of the Heisenberg commutator), and thus opens up the way for a systematic study of the semiclassical limit. Before publication, Dirac contrasts this work favorably to his own ideas on functional integration, in Bohr's Festschrift[Dir45], despite private reservations and lengthy arguments with Moyal.

M Bartlett and J Moyal (1949) [BM49] applies this language to calculate propagators and transition probabilities for oscillators perturbed by time-dependent potentials.

T Takabayasi (1954)[Tak54] investigates the fundamental projective normalization condition for pure state Wigner functions, and exploits Groenewold's link to the conventional density matrix formulation. It further illuminates the diffusion of wavepackets.

G Baker (1958)[Bak58] (Baker's thesis paper) envisions the logical autonomy of the formulation, predicated on the projective normalization condition as a basic postulate. It resolves measurement subtleties in the correspondence principle and appreciates the significance of the anticommutator of the $\star$-product as well, thus shifting emphasis to the $\star$-product itself, over and above its commutator.

D Fairlie (1964)[Fai64] (also see [Kun67,Coh76,Dah83]) explores the time-independent counterpart to Moyal's evolution equation, which involves the $\star$-product, beyond mere Moyal Bracket equations, and derives (instead of postulating) the projective orthonormality conditions for the resulting Wigner functions. These now allow for a unique and full solution of the quantum system, in principle (without any reference to the conventional Hilbert-space formulation). Autonomy of the formulation is fully recognized.

R Kubo (1964)[Kub64] elegantly reviews, in modern notation, the representation change between Hilbert space and phase space—although in ostensible ignorance of Weyl's and Groenewold's specific papers. It applies the phase-space picture to the description of electrons in a uniform magnetic field, initiating gauge-invariant formulations and pioneering "noncommutative geometry" applications to diamagnetism and the Hall effect.

N Cartwright (1976)[Car76] notes that the WF smoothed by a phase-space Gaussian as wide or wider than the minimum uncertainty packet is positive-semidefinite. Actually, the result goes further back to at least de Bruijn (1967)[deB67].

M Berry (1977)[Ber77] elucidates the subtleties of the semiclassical limit, ergodicity, integrability, and the singularity structure of Wigner function evolution.

F Bayen, M Flato, C Fronsdal, A Lichnerowicz, and D Sternheimer (1978)[BFF78] analyzes systematically the deformation structure and the uniqueness of the formulation, with special emphasis on spectral theory, and consolidates it mathematically. It provides explicit illustrative solutions to standard problems and utilizes influential technical tools, such as

the $\star$-exponential (already known in [Imr67,GLS68]).

A Royer (1977)[Roy77] interprets WFs as the expectation value of the operator effecting reflections in phase space. (Further see refs [Kub64,Gro76,BV94].)

G García-Calderón and M Moshinsky (1980)[GM80] implements the transition from Hilbert space to phase space to extend classical propagators and canonical transformations to quantum ones in phase space. (The most conclusive work to date is ref [BCW02]. Further see [HKN88,Hie82,DKM88,CFZ98,DV97,GR94,Hak99,KL99,DP01].)

J Dahl and M Springborg (1982)[DS82] initiates a thorough treatment of the hydrogen and other simple atoms in phase space, albeit not from first principles—the WFs are evaluated in terms of Schrödinger wave-functions.

M De Wilde and P Lecomte (1983)[deW83] consolidates the deformation theory of $\star$-products and MBs on general real symplectic manifolds, analyzes their cohomology structure, and confirms the absence of obstructions.

M Hillery, R O'Connell, M Scully, and E Wigner (1984)[HOS84] has done yeoman service to the physics community as the classic introduction to phase-space quantization and the Wigner function.

Y Kim and E Wigner (1990)[KW90] is a classic pedagogical discussion of the spread of wavepackets in phase space, uncertainty-preserving transformations, coherent and squeezed states.

B Fedosov (1994)[Fed94] initiates an influential geometrical construction of the $\star$-product on all symplectic manifolds.

T Curtright, D Fairlie, and C Zachos (1998)[CFZ98] illustrates more directly the equivalence of the time-independent $\star$-genvalue problem to the Hilbert space formulation, and hence its logical autonomy; formulates Darboux isospectral systems in phase space; works out the covariant transformation rule for general nonlinear canonical transformations (with reliance on the classic work of P Dirac (1933)[Dir33]); and thus furnishes explicit solutions of practical problems on first principles, without recourse to the Hilbert space formulation. Efficient techniques for perturbation theory are based on generating functions for complete sets of Wigner functions in T Curtright, T Uematsu, and C Zachos (2001)[CUZ01]. A self-contained derivation of the uncertainty principle in phase space is given in T Curtright and C Zachos (2001)[CZ01].

M Hug, C Menke, and W Schleich (1998)[HMS98] introduce and exemplify techniques for numerical solution of $\star$-equations on a basis of Chebyshev polynomials.

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