DEFORMATION QUANTIZATION SUPERINTEGRABILITY AND NAMBU MECHANICS

• Non-commutative geometry describes D-branes in a "magnetic" field background: different directions of space do not commute, $xy \neq yx$. The technical structure of NC geometry parallels that of **quantization in phase space**: $y \mapsto p$, $\theta \mapsto \hbar$. Quantum Mechanics' legacy to M-theory!

An equivalent alternative to Hilbert-space or path-integral quantization. Logically complete and self-standing (Weyl, Wigner, Moyal): one need not choose sides—coordinate or momentum space. It works in **full phase space**, accommodating the uncertainty principle. (Reviewed in Zachos, Int J Mod Phys **A17** (2002) 297-316 [hep-th/0110114])

The variables involved ("kernel functions" or "Weyl transforms of operators") are **c-number functions**, like those of the classical phase-space theory, and have the same interpretation, although they involve $\hbar$-corrections ("deformations"). It is only the detailed **algebraic structure** of their respective brackets and composition rules which contrast with the variables of the classical theory.

• Ordinary multiplication is supplanted by the cornerstone
noncommutative Star Product (Groenewold, 1946)

$$\boxed{f(x,p) \star g(x,p) \equiv f(x,p)\, e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)}\, g(x,p)} \, .$$

In practice, often evaluated through convolution,

$$f(\mathbf{r}) \star g(\mathbf{r}) = \left(\frac{\hbar}{\pi}\right)^2 \int d^2\mathbf{u}\, d^2\mathbf{v}\ f(\mathbf{r}+\mathbf{u}) g(\mathbf{r}+\mathbf{v})\, e^{\frac{2i}{\hbar}\ \mathbf{u}\times\mathbf{v}}\ .$$

A noncommutative, associative, pseudodifferential operation.

• Encodes the entire quantum mechanical action.

• Its antisymmetrization (commutator) is the Moyal Bracket (MB $\{\{,\}\}$).

Instead of a wavefunction, one solves for the Wigner Function (WF), the kernel function of the density matrix, which is a quasi-probability distribution function in phase-space,

$$f_{mn}(x,p) = \int dy\ \psi_m^*(x - \frac{\hbar}{2}y)\ e^{-iyp}\ \psi_n(x + \frac{\hbar}{2}y).$$

$\leadsto$ Observables and transition amplitudes are phase-space integrals of kernel functions weighted by the WF, in analogy to statistical mechanics.

$$\frac{\partial f}{\partial t} = \{\{H, f\}\}\ .$$

• WFs obey quasi-orthonormality and completeness relations; and, characteristically, nonlocal differential eigenvalue equations (analogous to Schrödinger's equation), eg:

$$H \star f_{mn} = H\left(x + \frac{i\hbar}{2}\overrightarrow{\partial}_p\ ,\ p - \frac{i\hbar}{2}\overrightarrow{\partial}_x\right)\ f_{mn}(x,p) = E_n\ f_{mn}(x,p).$$

But... WHY BOTHER??

• Can quantize superintegrable systems maximally *preserving the symmetry of the classical system*—when there would be operator ordering ambiguities in conventional QM (eg, velocity/momentum-dependent potentials).

eg, σ-MODELS

$$\boxed{\{I,H\}=0, \qquad \rightsquigarrow \qquad \{\{I_{qm},H_{qm}\}\} \equiv \frac{I_{qm}\star H_{qm}-H_{qm}\star I_{qm}}{i\hbar}=0}.$$

(As $\hbar \rightarrow 0$, MB $\rightarrow$ PB.)

• We find $O(\hbar^2)$ corrections in the H_{qm}s but **not** the I_{qm}s.

$$L(q,\dot{q})=\frac{1}{2}g_{ab}(q)\,\dot{q}^a\dot{q}^b, \qquad \rightsquigarrow$$

$$p_a=\frac{\partial L}{\partial \dot{q}^a}=g_{ab}\,\dot{q}^b, \qquad\qquad \dot{q}^a=g^{ab}p_b\;.\rightsquigarrow$$

$$H(p,q)=\frac{1}{2}g^{ab}p_ap_b \qquad (=L).$$

$$\dot{p}_a=-\frac{g^{bc}_{,a}}{2}\,p_bp_c=\frac{g_{bc,a}}{2}\,\dot{q}^b\dot{q}^c.$$

eg, S^2 (Schrödinger, Velo & Wess, Higgs...):
Eliminate z, so $q^1=x, q^2=y$, $a,b=1,2$.

$$g_{ab}=\delta_{ab}+\frac{q^aq^b}{u}, \qquad g^{ab}=\delta_{ab}-q^aq^b, \qquad \det g_{ab}=\frac{1}{u}, \qquad u\equiv 1-x^2-y^2\;.$$

$$p_a=\dot{q}^a+q^a\frac{h}{u}=\dot{q}^a+q^a(q\cdot p)\;, \qquad\qquad h\equiv-\dot{u}/2=x\dot{x}+y\dot{y}\;.$$

$$\dot{p}_a=p_a\,q\cdot p\;, \qquad \text{ie,} \qquad \ddot{q}^{\,a}+q^a\left(\frac{\dot{h}}{u}+\frac{h^2}{u^2}\right)=0.$$

The isometries of the manifold generate the conserved integrals of the motion: three classical invariants

$$L_z=xp_y-yp_x\;, \qquad L_y=\sqrt{u}\,p_x\;, \qquad L_x=-\sqrt{u}\,p_y\;.$$

PBs close into $SO(3)$,

$$\{L_x, L_y\} = L_z \ , \qquad \{L_y, L_z\} = L_x \ , \qquad \{L_z, L_x\} = L_y \ .$$

$\rightsquigarrow$ PBs with the Casimir invariant $\mathbf{L} \cdot \mathbf{L}$ vanish.
Since $H = \mathbf{L} \cdot \mathbf{L}/2$, they are manifested to be time-invariant,

$$\dot{\mathbf{L}} = \{\mathbf{L}, H\} = 0.$$

Quantize by insertion of $\star$s at strategic points and orderings of the variables to maintain maximal integrability,

$$\boxed{H_{qm} = \tfrac{1}{2}\left(L_x \star L_x + L_y \star L_y + L_z \star L_z\right)}.$$

The reason: in this realization, the algebra is promoted to the corresponding MB expression **without any modification**, since all of its MBs collapse to PBs: all corrections $O(\hbar)$ vanish.
$\rightsquigarrow$ these particular invariants are undeformed by quantization,
$\mathbf{L} = \mathbf{L}_{qm}$. $\rightsquigarrow$ given associativity for $\star$,

$$\{\!\{\mathbf{L} \cdot \star\mathbf{L}, \ \mathbf{L}\}\!\} = 0.$$

• Quantum correction:

$$\boxed{H_{qm} = H + \tfrac{\hbar^2}{8}(\det g - 3)}.$$

• Spectrum $\propto$ $\hbar^2 l(l+1)$, the spectrum of the $SO(3)$ Casimir $\mathbf{L}\cdot\star\mathbf{L} = L_+\star L_- + L_z\star L_z - \hbar L_z$, for integer l.

• Produced algebraically by the **identical** standard recursive ladder operations in $\star$ space which obtain in the operator formalism Fock space,

$$L_z\star L_+ - L_+\star L_z = \hbar L_+ \ ,$$

where $L_\pm \equiv L_x \pm i L_y$.

Must also bound the $\star$-spectrum of L_z: From the real $\star$-square theorem (TC & CZ),

$$\langle \mathbf{L}\cdot\star\mathbf{L} - L_z\star L_z\rangle = \langle L_x\star L_x + L_y\star L_y\rangle \geq 0 \ .$$

The $\star$-genvalues of L_z, m, are thus bounded, $|m| \leq l < \sqrt{\langle \mathbf{L}\cdot\star\mathbf{L}\rangle}/\hbar$, necessitating $L_-\star f_{m=-l} = 0$. $\rightsquigarrow$

$$L_+\star L_-\star f_{-l} = 0 = (\mathbf{L}\cdot\star\mathbf{L} - L_z\star L_z + \hbar L_z)\star f_{-l} \ ,$$

$\rightsquigarrow$

$$\langle \mathbf{L}\cdot\star\mathbf{L}\rangle = \hbar^2 l(l+1) \ .$$

CHIRAL MODELS (eg, S^3)

Geometrical advantages due to chiral structure $G \otimes G$. Vielbeine, $g_{ab} = \delta_{ij} V^i_a V^j_b$ and $g^{ab} V^i_a V^j_b = \delta^{ij}$.
Dreibeine are either left-invariant, or right-invariant,

$$^{(\pm)}V^i_a = \epsilon^{iab} q^b \pm \sqrt{u}\, g_{ai} \,, \qquad\qquad ^{(\pm)}V^{ai} = \epsilon^{iab} q^b \pm \sqrt{u}\, \delta^{ai}.$$

The corresponding right and left conserved charges (left- and right-invariant, respectively)

$$R^i = {}^{(+)}V^i_a \, \dot{q}^a = {}^{(+)}V^{ai} p_a \,, \qquad\qquad L^i = {}^{(-)}V^i_a \, \dot{q}^a = {}^{(-)}V^{ai} p_a \,,$$

or linear combinations into Axial and Isospin charges (again linear in the momenta),

$$\frac{\mathbf{R} - \mathbf{L}}{2} = \sqrt{u}\, \mathbf{p} \equiv \mathbf{A} \,, \qquad \frac{\mathbf{R} + \mathbf{L}}{2} = \mathbf{q} \times \mathbf{p} \equiv \mathbf{I} \,.$$

The **L**s and the **R**s have PBs closing into standard $SU(2) \otimes SU(2)$, ie, $SU(2)$ relations within each set, and vanishing between the two sets.
$\rightsquigarrow$

$$H = \frac{1}{2} \mathbf{L} \cdot \mathbf{L} = \frac{1}{2} \mathbf{R} \cdot \mathbf{R} = L \,.$$

The quantum invariants **L** and **R** again coincide with the classical ones, without deformation. Eigenvalues of the relevant Casimir invariant now $j(j+1)$, for half-integer j.

• The symmetric quantum hamiltonian is **simpler** than for the 2-sphere (and other N-spheres): it can now be also written **geometrically**,

$$\boxed{H_{qm} = \tfrac{1}{2} (p_a V^{ai}) \star (V^{bi} p_b) = \tfrac{1}{2} \left(g^{ab} p_a p_b + \tfrac{\hbar^2}{4} \partial_a V^{bi} \partial_b V^{ai} \right)} \,.$$

• Quantum correction

$$H_{qm} - H = \frac{\hbar^2}{8}(\det g - 7) = \frac{\hbar^2}{8}\left(\frac{1}{1-q^2} - 7\right) .$$

(In operator language, for **operators** $\mathfrak{x}$ and $\mathfrak{p}$), it would appear more complex: in the Weyl correspondence, the first term, $g^{ab}(x)p_a p_b/2 \quad \mapsto$

$$\frac{1}{8}\left(\mathfrak{p}_a \mathfrak{p}_b g^{ab}(\mathfrak{x}) + 2\mathfrak{p}_a g^{ab}(\mathfrak{x})\mathfrak{p}_b + g^{ab}(\mathfrak{x})\mathfrak{p}_a\mathfrak{p}_b\right) = \frac{1}{2}\mathfrak{p}_a g^{ab}(\mathfrak{x})\mathfrak{p}_b + \frac{3\hbar^2}{4} ,$$

while the second term would be the unambiguous.)

• In general,

$$iU^{-1}\frac{d}{dt}U = {}^{(+)}V_a^j T_j \dot{q}^a = {}^{(+)}V^{aj} p_a T_j , \qquad iU\frac{d}{dt}U^{-1} = {}^{(-)}V^{aj} p_a T_j ,$$

$\rightsquigarrow$ The PBs of the left- and right-invariant charges ${}^{(\pm)}V^{aj}p_a = \frac{i}{2} Tr T_j U^{\mp 1}\frac{d}{dt}U^{\pm 1}$ close to the identical Lie algebras,

$$\{{}^{(\pm)}V^{aj}p_a, {}^{(\pm)}V^{bk}p_b\} = -2f^{jkn}\, {}^{(\pm)}V^{an}p_a ,$$

and PB commute with each other,

$$\{\, {}^{(+)}V^{aj}p_a, {}^{(-)}V^{bk}p_b\} = 0.$$

$\rightsquigarrow$

$$\boxed{H_{qm} - H = \tfrac{\hbar^2}{8}\left(\Gamma^b_{ac}\, g^{cd}\Gamma^a_{bd} - f_{ijk}f_{ijk}\right)} .$$

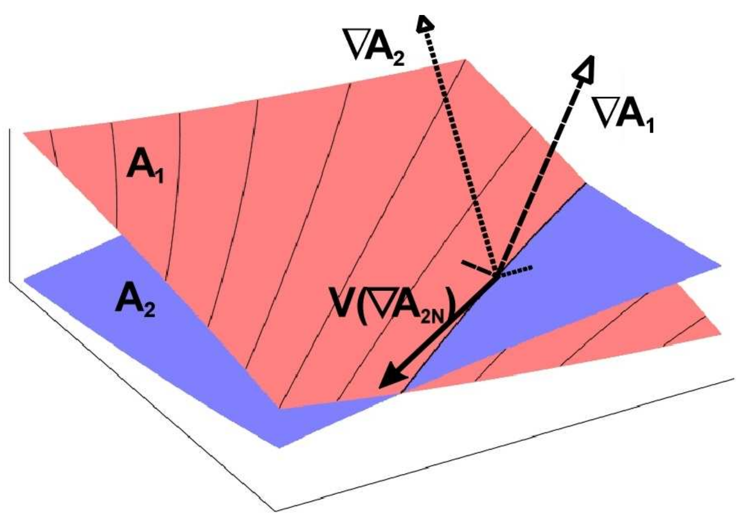
• N-spheres not as geometrically elegant.

$$H_{qm} - H = \frac{\hbar^2}{8}\left(\frac{1}{u} - 1 - N(N-1)\right) .$$

Spectra proportional to the quadratic Casimir eigenvalues $l(l+N-1)$ for integer l.

Extra invariants beyond those required for integrability.
Optimal, elegant accounting through NBs in phase space.
In an N-dimensional space, $\rightsquigarrow$ $2N$-dimensional phase space, motion is confined on the constant surfaces specified by the algebraically independent integrals of the motion (eg, L_x, L_y, L_z for S^2.)

$\rightsquigarrow$ the phase-space velocity $\mathbf{v} = (\dot{\mathbf{q}}, \dot{\mathbf{p}})$ is always perpendicular to the $2N$-dim **phase space gradients** $\nabla = (\partial_{\mathbf{q}}, \partial_{\mathbf{p}})$ of all these integrals of the motion.



• If there are $2N-1$ algebraically independent integrals, the phase-space velocity must be proportional to the cross-product of **all** those gradients.

$\rightsquigarrow$ Motion fully specified for any phase-space function $k(\mathbf{q},\mathbf{p})$ by a phase-space Jacobian,

$$\frac{dk}{dt} = \nabla k \cdot \mathbf{v}$$

$$
\begin{aligned}
&\propto \ \partial_{i_1} k \, \epsilon^{i_1 i_2 \ldots i_{2N}} \ \partial_{i_2} L_{i_1} \ldots \partial_{i_{2N}} L_{2N-1} \\
&= \ \frac{\partial(k, L_1, \ldots, \ldots, L_{2N-1})}{\partial(q_1, p_1, q_2, p_2, \ldots, q_N, p_N)} \\
&\equiv \ \{k, L_1, \ldots, L_{2N-1}\} \ .
\end{aligned}
$$

$\equiv$ the Nambu Bracket

- The proportionality constant is shown to be time-invariant.

Superintegrable systems in phase space cannot **avoid** being described by NBs.

eg, S^2:

$$
\boxed{\frac{dk}{dt} = \frac{\partial\left(k, L_x, L_y, L_z\right)}{\partial\left(x, p_x, y, p_y\right)}} \ .
$$

eg, S^N:

$$
\frac{dk}{dt} = \frac{(-1)^{\left(N^2-1\right)}}{P_2 P_3 \cdots P_{N-1}} \frac{\partial\left(k, P_1, L_{12}, P_2, L_{23}, P_3, \cdots, P_{N-1}, L_{N-1\ N}, P_N\right)}{\partial\left(x_1, p_1, x_2, p_2, \cdots, x_N, p_N\right)} ,
$$

($P_a = \sqrt{u}\, p_a$, for $a = 1, \cdots, N$, and $L_{a,a+1} = q^a p_{a+1} - q^{a+1} p_a$, for $a = 1, \cdots, N-1$.)

In general NBs possess all antisymmetries of Jacobian determinants; and obey the Leibniz rule,

$$
\{k(L, M), f_1, f_2, \ldots\} = \frac{\partial k}{\partial L}\{L, f_1, f_2, \ldots\} + \frac{\partial k}{\partial M}\{M, f_1, f_2, \ldots\}.
$$

$\rightsquigarrow$ Eg, the hamiltonian is constant,

$$
\frac{dH}{dt} = \left\{\frac{\mathbf{L}\cdot\mathbf{L}}{2}, \ldots, \ldots, \ldots\right\} = 0,
$$

• The impossibility to antisymmetrize more than $2N$ indices in $2N$-dimensional phase space,

$$\epsilon^{ab....c[i}\epsilon^{j_1 j_2 \cdots j_{2N}]} = 0 \ ,$$

leads to the (generalized) "Fundamental" Identity (FI),

$$\{f_0\{f_1,...,f_{m-1},f_m\},f_{m+1},...,f_{2m-1}\} + \{f_m,f_0\{f_1,...,f_{m-1},f_{m+1}\},f_{m+2},...,f_{2m-1}\}$$
$$+... + \{f_m,...,f_{2m-2},f_0\{f_1,...,f_{m-1},f_{2m-1}\}\} = \{f_1,...,f_{m-1},f_0\{f_m,f_{m+1},...,f_{2m-1}\}\}.$$

not **the** generalization of the Jacobi Identity—more like a consequence of a derivation property.

Closure under PBs of quantities serving as arguments in the NB does *not* suffice for a NB to vanish: viz. $\{L_x, L_y\} = L_z$. But it is always true that PBs of conserved integrals are themselves conserved integrals:

$$\frac{d\{L_a, L_b\}}{dt} \propto \{\{L_a, L_b\}, L_1, ..., L_{2N-1}\}$$

must vanish.

• PBs result from a maximal reduction of NBs, by inserting $2N-2$ phase-space coordinates and summing over them, thereby taking **symplectic traces,**

$$\{L,M\} = \frac{1}{(N-1)!}\left\{L,M,x_{i_1},p_{i_1},\cdots,x_{i_{N-1}},p_{i_{N-1}}\right\} .$$

• Fewer traces lead to relations between NBs of maximal rank, $2N$, and those of lesser rank, $2k$,

$$\{L_1,\cdots,L_{2k}\} = \frac{1}{(N-k)!}\left\{L_1,\cdots,L_{2k},x_{i_1},p_{i_1},\cdots,x_{i_{N-k}},p_{i_{N-k}}\right\} .$$

Essentially, $\{L_1,\cdots,L_{2k}\}$ acts like a Dirac Bracket (DB), up to a normalization $\{L_1,L_2\}_{DB}$. The fixed additional entries $L_3,\cdots,L_{2k}$ in the NB play the role of the constraints in the DB.

$\rightsquigarrow$ DB satisfies the Jacobi Identity.

• By virtue of this symplectic trace, for a general system—not only a superintegrable one—Hamilton's equations admit a different NB expression,

$$\frac{dk}{dt} = \{k,H\} = \frac{1}{(N-1)!}\{k,H,x_{i_1},p_{i_1},...,x_{i_{N-1}},p_{i_{N-1}}\} .$$

• More elaborate isometries of general manifolds in such models expected to yield to similar analysis.

QUANTIZATION of NBs (Nambu vs Zariski)

Undeserved bad reputation, on account of top-down shortcomings. But, in any case ....

• It **must** coincide with Moyal, or standard, quantization for the specific models above! Does it?

Nambu's (1973) proposal (here applied to phase-space), QNBs:

$$[A,B]_\star \equiv i\hbar \, \{\{A,B\}\}$$

$$[A,B,C]_\star \equiv A\star B\star C - A\star C\star B + B\star C\star A - B\star A\star C + C\star A\star B - C\star B\star A$$

$$[A,B,C,D]_\star \equiv A\star[B,C,D]_\star - B\star[C,D,A]_\star + C\star[D,A,B]_\star - D\star[A,B,C]_\star =$$

$$= [A,B]_\star\star[C,D]_\star + [A,C]_\star\star[D,B]_\star + [A,D]_\star\star[B,C]_\star$$

$$+[C,D]_\star\star[A,B]_\star + [D,B]_\star\star[A,C]_\star + [B,C]_\star\star[A,D]_\star \, .$$

Full antisymmetry, but no Leibniz property or FI, **in general**. Only a subjective shortcoming, dependent on the specific application context! **Quantization is consistent**.

Objectively, for S^2,

$$\boxed{\frac{dk}{dt} = \{\{k, H_{qm}\}\} = \frac{-1}{2\hbar^2}\,[k, L_X, L_Y, L_Z]_\star} \, ,$$

a derivation. $\rightsquigarrow$ **Here**, in phase space, Leibniz and FI hold, **nevertheless**. Good $\hbar \to 0$ limit.

NB. For constant A, thus $dA/dt = 0$, $[A,B,C,D]_\star = 0$ holds identically, in contrast to the 3-argument QNB. Thus, **no debilitating constraint** among the arguments B, C, D is imposed; the inconsistency identified originally is a feature of odd-argument QNBs, and does not restrict the even-argument QNBs of phase space.

• By contrast, one might try to define a quantized Nambu bracket $\{\{,,,\}\}_\bigstar$ by taking $\star$-products of the phase-space gradients that appear in the classical NB and applying Jordan's trick of symmetrizing all such products, at the expense of making the algebra non-associative. (Also fails to grant all 3 wishes of mathematicians: antisymmetry, Leibniz rule, and FI).
But it moreover **does not** give the correct quantum equations of motion.

• More generic situation, eg for S^N, $N>2$: the QNBs provide the correct quantization rule, but **need not** satisfy the naive Leibniz property (and FI) for consistency, as they are not necessarily plain derivations, but time derivatives are entwined inside strings of invariants. Eg, for S^3,

$$[k,P_1,L_{12},P_2,L_{23},P_3]_\star = 3i\hbar^3(P_2\star\{\{k,H_{qm}\}\}+\{\{k,H_{qm}\}\}\star P_2)+\mathcal{Q}(O(\hbar^5)).$$

$\rightsquigarrow$

$$[k,P_1,L_{12},P_2,L_{23},P_3]_\star = 3i\hbar^3\frac{d}{dt}(P_2\star k+k\star P_2)+\mathcal{Q}(O(\hbar^5)).$$

The right hand side is not an unadorned derivation on k
$\rightsquigarrow$ does not impose a Leibniz rule on the left hand side. (Other consistencey constraints are more suitable and are satisfied.)

• [hep-th/0212267] Quantum NBs are consistent and describe the quantum behavior of superintegrable systems equivalently to standard hamiltonian quantization. All reputed inconsistencies have been addressing unsuitable (and untenable) conditions.