The NRQCD Factorization Approach

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Factorization: A Separation of Scales

- In heavy-quarkonium decays and hard-scattering production, large scales appear:
  Both the heavy-quark mass m and p_T are much larger than Λ_{QCD}.
- Hope: Because of the large scales, asymptotic freedom will allow us to do perturbation theory.

$$\alpha_s(m_c) \approx 0.25; \qquad \alpha_s(m_b) \approx 0.18.$$

- But there are clearly low-momentum, nonperturbative effects in the heavy-quarkonium dynamics.
- We wish to separate the short-distance/high-momentum, perturbative effects from the long-distance/low-momentum, nonperturbative effects.
- This separation is known as “factorization.”

- Many factorization procedures in particle theory can be understood simply in terms of effective field theories.
- Examples are
  - the operator-product expansion (OPE) for deep-inelastic scattering, τ decays, $R(e^+e^-)$,
  - heavy-quark effective theory (HQET) for heavy-light-meson decays,
  - heavy-quarkonium decays.
- Factorization proofs for hadron-hadron induced reactions require (so far) all-orders perturbative arguments.
- Examples are
  - large-P_T hadron-hadron scattering,
  - Drell-Yan production of lepton pairs,
  - heavy-quarkonium production.
- Even in the hadron-hadron case, effective-field theory methods (SCET) apply to the individual hadrons.

Heavy-Quarkonium: A Multi-Scale Problem

- In analyzing heavy-quarkonium (spectrum, decay rates, production rates), we would like to separate contributions at the perturbative scale m from those at smaller, nonperturbative scales.
- In quarkonium, the next scale below m is heavy-quark momentum $mv \sim 1/R$, where v is the heavy-quark velocity in the bound state.
  - $v^2 \approx 0.3$ for charmonium.
  - $v^2 \approx 0.1$ for bottomonium.
- The heavy-quark energy $mv^2/2$ and $\Lambda_{\rm QCD}$ are also important scales.
- Potential NRQCD (pNRQCD) (Brambilla, Pineda, Soto, Vairo) separates the scale mv from the smaller scales.
- Here our goal is only to separate the the scale m from the smaller scales.
- Construct an effective field theory valid at scales mv and smaller by integrating out the high-momentum contributions.

Nonrelativistic QCD (NRQCD)

- Generalization of NRQED (W. E. Caswell, G. P. Lepage).
- The effective theory has a UV cutoff $\Lambda \sim m$.
- For processes with $p < \Lambda$, the effective theory reproduces full QCD.
- Processes with $p > \Lambda$ are not described in the effective theory, but they affect the coefficients of local interactions.

Gedanken Construction of NRQCD

- In the path integrals for the amplitudes in QCD, integrate out:
  all light-quark and gluon modes with $p > \Lambda$,
  all the heavy-quark modes with $E - m,\ |\mathbf{p}| > \Lambda$.
- For the gluon–light-quark sector, the effective action is a cut-off version of the full action (e.g. lattice) plus "improvement" terms.

- For the heavy-quark sector, carry out a unitary transformation to diagonalize the interactions in term of the Q and $\bar{Q}$ parts of the Dirac field. For example, in full QCD the Foldy-Wouthuysen transformation

$$\Psi \to \exp[-i\gamma \cdot \boldsymbol{D}/(2m)]\Psi$$

leads to an approximate action

$$\begin{pmatrix} \psi \\ \chi \end{pmatrix}^\dagger \begin{pmatrix} -m + iD_t + \frac{\boldsymbol{D}^2}{2m} & 0 \\ 0 & m + iD_t - \frac{\boldsymbol{D}^2}{2m} \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix},$$

which is correct up to terms of relative order $p^2/m^2 \sim v^2$.

  - ψ is the Pauli spinor field that annihilates a heavy quark.
  - χ is the Pauli spinor field that creates a heavy anti-quark.

- Remove m and any energy shift ΔE from the Q and $\bar{Q}$ energies at zero momentum by transforming the fields by a phase:

$$\begin{aligned} \psi &\to e^{-i(m+\Delta E)t}\psi \\ \chi &\to e^{i(m+\Delta E)t}\chi. \end{aligned}$$

- In practice, this procedure would be very complicated.

- Instead, write down all possible interactions that are consistent with the symmetries of full QCD.

Symmetries of (NR)QCD

- $SU(3)$ gauge symmetry
  Gluon fields enter the effective action only through D_0, $\boldsymbol{D}$, $\boldsymbol{E}$, and $\boldsymbol{B}$.
- Rotational symmetry
  Remaining subgroup of Lorentz symmetry.
- Charge conjugation

$$\begin{aligned} \psi &\longrightarrow -i\sigma_2(\chi^T)^\dagger, \\ \chi &\longrightarrow i\sigma_2(\psi^T)^\dagger. \end{aligned}$$

- Parity

$$\begin{aligned} \psi(t,\boldsymbol{r}) &\longrightarrow \psi(t,-\boldsymbol{r}), \\ \chi(t,\boldsymbol{r}) &\longrightarrow -\chi(t,-\boldsymbol{r}). \end{aligned}$$

- In addition, impose heavy-quark phase symmetry.
  Separate quark and antiquark number conservation.

$$\begin{aligned} \psi &\longrightarrow e^{i\alpha}\psi, \\ \chi &\longrightarrow e^{i\beta}\chi. \end{aligned}$$

NRQCD Action

- Leading terms in $p/m = v$ are just the Schrödinger action.

$$\mathcal{L}_{\mathrm{heavy}} = \psi^\dagger \left(iD_t + \frac{\mathbf{D}^2}{2m} \right) \psi + \chi^\dagger \left(iD_t - \frac{\mathbf{D}^2}{2m} \right) \chi$$

$$D_t = \partial_t + igA_0 .$$

$$\mathbf{D} = \partial - ig\mathbf{A} .$$

- ψ is the Pauli spinor field that annihilates a heavy quark.
- χ is the Pauli spinor field that creates a heavy antiquark.

- To reproduce QCD completely, we would need an infinite number of interactions. For example,

$$
\begin{aligned}
\delta\mathcal{L}_{\text{bilinear}} \; = \; & \frac{c_1}{8m^3}\left[\psi^\dagger(\mathbf{D}^2)^2\psi \; - \; \chi^\dagger(\mathbf{D}^2)^2\chi\right] \\
& + \; \frac{c_2}{8m^2}\left[\psi^\dagger(\mathbf{D}\cdot g\mathbf{E} - g\mathbf{E}\cdot\mathbf{D})\psi \right. \\
& \qquad\qquad \left. + \; \chi^\dagger(\mathbf{D}\cdot g\mathbf{E} - g\mathbf{E}\cdot\mathbf{D})\chi\right] \\
& + \; \frac{c_3}{8m^2}\left[\psi^\dagger(i\mathbf{D}\times g\mathbf{E} - g\mathbf{E}\times i\mathbf{D})\cdot\boldsymbol{\sigma}\psi \right. \\
& \qquad\qquad \left. + \; \chi^\dagger(i\mathbf{D}\times g\mathbf{E} - g\mathbf{E}\times i\mathbf{D})\cdot\boldsymbol{\sigma}\chi\right] \\
& + \; \frac{c_4}{2m}\left[\psi^\dagger(g\mathbf{B}\cdot\boldsymbol{\sigma})\psi \; - \; \chi^\dagger(g\mathbf{B}\cdot\boldsymbol{\sigma})\chi\right].
\end{aligned}
$$

- In practice, work to a given precision in v.
- The c_i are called short-distance coefficients.
  - They can be computed in perturbation theory by matching amplitudes in full QCD and NRQCD.
  - By design, all of the low-scale physics is contained in the explicit NRQCD interactions.
  - The c_i contain the effects from momenta $> \Lambda$.
- Λ plays the rôle of a factorization scale between the hard and soft physics.

Elimination of Time Derivatives in the NRQCD Action

- Use field re-definitions to eliminate higher-order terms in which D_t acts on ψ or χ.
- Under variations in $\psi^\dagger$ and $\chi^\dagger$,

$$\delta\mathcal{L} \approx \delta\psi^\dagger \left(iD_t + \frac{\boldsymbol{D}^2}{2m} \right) \psi + \delta\chi^\dagger \left(iD_t - \frac{\boldsymbol{D}^2}{2m} \right) \chi.$$

- If there are terms in the action

$$\psi^\dagger G(iD_t)\psi,$$
$$\chi^\dagger G'(iD_t)\chi,$$

choose

$$\begin{aligned} \delta\psi^\dagger &= -\psi^\dagger G, \\ \delta\chi^\dagger &= -\chi^\dagger G'. \end{aligned}$$

The effect is

$$\begin{aligned} iD_t\psi &\to \left(\frac{-\boldsymbol{D}^2}{2m} \right) \psi, \\ iD_t\chi &\to \left(\frac{\boldsymbol{D}^2}{2m} \right) \chi. \end{aligned}$$

- Changes amount to use of the equations of motions: On-shell amplitudes are unaffected.
- Start with terms in the action of lowest order in v and proceed iteratively through terms of the desired accuracy.

Matching

- Determine c_i's by matching amplitudes on shell.
- Required because of the use of field re-definitions (equations of motion).
- Convenient because it makes the matching gauge invariant.
- The short-distance coefficients are independent of the $Q\bar{Q}$ state:
  Use free $Q\bar{Q}$ states to do the matching in perturbation theory.
- Example: tree-level matching of the pole in the heavy-quark propagator.
  Full QCD gives

$$E = \sqrt{\boldsymbol{p}^2 + m_Q^2} - m_Q = \frac{\boldsymbol{p}^2}{2m_Q} - \frac{\boldsymbol{p}^4}{8m_Q^3} + \cdots .$$

NRQCD gives

$$E = \frac{\boldsymbol{p}^2}{2m} - c_1\frac{\boldsymbol{p}^4}{8m^3} + \cdots .$$

Fixes $m = m_Q$ and $c_1 = 1$.

Counting Powers of v

- Normalization:

$$\int d^3x\, \psi^\dagger(x)\psi(x) = 1.$$

- For a bound state of size $\sim 1/p$,

$$\int d^3x \sim 1/p^3.$$

- Implies that

$$\psi^\dagger(x)\psi(x) \sim p^3.$$

- Since the typical wave-function momentum is p,

$$\boldsymbol{\nabla}\psi(x) \sim p\psi(x).$$

- Consistency of the equations of motion can be used to determine the v scaling of other operators.
- Gives the same result as an all-orders perturbative analysis, but valid beyond perturbation theory.
- Specialize to Coulomb gauge.
  - We will see that in Coulomb gauge $\boldsymbol{A}$ is small compared with $A_0 \equiv \phi$.
- Lowest-order equation for the quark field:

$$\left(i\partial_t - g\phi(x) + \frac{\boldsymbol{\nabla}^2}{2m}\right)\psi(x) = 0.$$

The virial theorem for a bound state implies that

$$\partial_t \psi \sim g\phi\psi \sim \frac{\boldsymbol{\nabla}^2}{2m}\psi \sim mv^2\psi.$$

- Implies that

$$g\phi \sim mv^2.$$

- The lowest-order equation for ϕ:

$$\boldsymbol{\nabla}^2 g\phi(x) = -g^2\psi^\dagger(x)\psi(x).$$

- Assume that the gluon field has momentum of order $p = mv$.
  - True for binding gluons, but not necessarily for others.
  - Will consider Fock-state gluons with momentum of order mv^2 later.
- Implies that

$$g\phi(x) \sim \frac{1}{p^2} g^2 p^3 \sim g^2 p.$$

- This is consistent with the previous estimate iff

$$\alpha_s(mv) \sim g^2(mv) \sim v.$$

- The lowest-order equation for $\boldsymbol{A}$:

$$(\partial_t^2 - \boldsymbol{\nabla}^2) g\boldsymbol{A} = (g^2/m)\psi^\dagger \boldsymbol{\nabla}\psi + g\phi\boldsymbol{\nabla} g\phi.$$

  If we assume that the momentum of $\boldsymbol{A}$ is $\sim p$, this implies that

$$g\boldsymbol{A}(x) = \frac{1}{p^2}\left(\frac{g^2}{m}p^4 + p(mv^2)^2\right) \sim mv^3$$

- $g\boldsymbol{A}$ is smaller than $g\phi$ by a factor v.
- If we again assume that ϕ and $\boldsymbol{A}$ have momenta of order p, it follows immediately that

$$g\boldsymbol{E} \approx -\boldsymbol{\nabla} g\phi \sim m^2 v^3,$$

$$g\boldsymbol{B} \approx \boldsymbol{\nabla} \times g\boldsymbol{A} \sim m^2 v^4.$$

Operator	Estimate
α_s	v
ψ	$(mv)^{3/2}$
χ	$(mv)^{3/2}$
D_t (acting on ψ or χ)	mv^2
$\mathbf{D}$ (acting on ψ or χ)	mv
$g\mathbf{E}$	m^2v^3
$g\mathbf{B}$	m^2v^4
$g\phi$ (in Coulomb gauge)	mv^2
$g\mathbf{A}$ (in Coulomb gauge)	mv^3

Heavy-Quark Spin Symmetry

- The velocity-scaling rules tell us that, in the action, the leading spin-flip terms

$$\psi^\dagger(g\mathbf{B}\cdot\boldsymbol{\sigma})\psi \;-\; \chi^\dagger(g\mathbf{B}\cdot\boldsymbol{\sigma})\chi$$

  are suppressed by v^2 compared with the leading non-spin-flip terms

$$\psi^\dagger\left(iD_t+\frac{\mathbf{D}^2}{2m}\right)\psi \;+\; \chi^\dagger\left(iD_t-\frac{\mathbf{D}^2}{2m}\right)\chi.$$

- Up to corrections of order v^2:
  - The spin parts of wave functions factor from the non-spin parts.
  - If two quarkonium states are related by a spin flip, their energies and the non-spin parts of their wave functions are the same.
- This leads to relations between matrix elements of operators in quarkonium states that are related by a spin flip. (Examples later.)

Probability to Emit a Gluon from a Quarkonium State

- The leading non-spin-flip heavy-quark–spatial-gluon vertex [from $\psi^\dagger(\mathbf{D}^2/2m)\psi$ in the action]:

$$\frac{ig(2p+k)_i}{2m} \sim gv.$$

  p is the heavy-quark momentum, which is of order mv.
  k is the gluon momentum, which can be of order mv or mv^2.

- The heavy-quark propagator:

$$\frac{i}{E+k-(\mathbf{p}+\mathbf{k})^2/(2m)} \sim \frac{1}{k}.$$

  E is the heavy-quark energy, which is of order mv^2.

- Amplitude to emit a gluon that doesn't flip the spin: $\sim gv/k$.

- The probability is $(\text{amplitude})^2 \times$ phase space:

$$P_{\text{non-flip}} \sim \int \frac{d^3k}{2k}\left(\frac{gv}{k}\right)^2 \sim g^2v^2 \sim \begin{cases} v^3 & \text{for } k \sim mv; \\ v^2 & \text{for } k \sim mv^2. \end{cases}$$

  We assume that $g^2(mv) \sim v$ and $g^2(mv^2) \sim v^0$.

- The leading spin-flip heavy-quark–spatial-gluon vertex [from $\psi^\dagger (g\boldsymbol{\sigma} \cdot \mathbf{B}/2m)\psi$ in the action]:

$$g\epsilon_{ijl}k_j\sigma_l \sim g\frac{k}{m}.$$

- The propagator is still $\sim 1/k$.
- The amplitude to emit a gluon that flips the spin:

$$\sim (gk/m)/k \sim g/m.$$

- The probability to emit a gluon that flips the spin:

$$P_{\text{flip}} \sim \int \frac{d^3k}{2k}\left(\frac{g}{m}\right)^2 \sim \frac{g^2k^2}{m^2} \sim \begin{cases} v^3 & \text{for } k \sim mv; \\ v^4 & \text{for } k \sim mv^2. \end{cases}$$

- Conclusion:
  - Emission of a gluon that doesn't flip the spin costs a factor v^2 in probability.
  - Emission of a gluon that flips the spin costs a factor v^3 in probability.
- This result can probably be proven to all orders in perturbation theory.

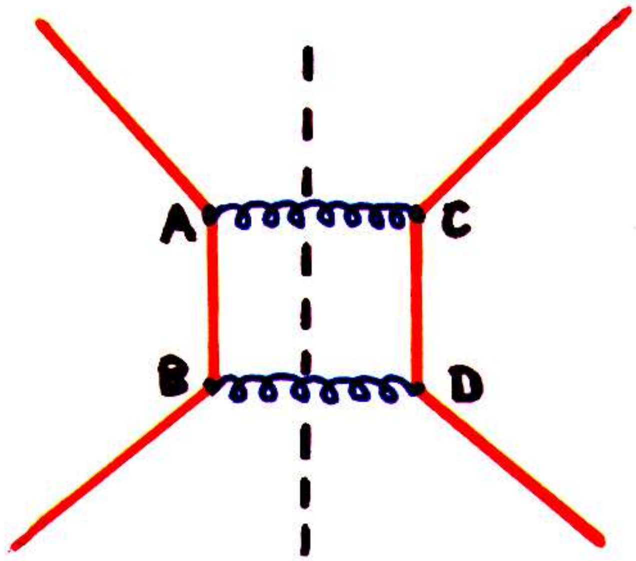
Fock-State Expansion

- The $Q\bar{Q}$ Fock state $|Q\bar{Q}\rangle$ is leading in v.
  - It has the quantum numbers of the quarkonium.
- There are subleading Fock states such as $|Q\bar{Q}g\rangle$, $|Q\bar{Q}gg\rangle$, $|Q\bar{Q}q\bar{q}\rangle$.
  - In the subleading Fock states, the $Q\bar{Q}$ pair can have different spin, orbital-angular momentum, and color than the quarkonium.
- The subleading Fock states are suppressed by a probability factor
  - v^2 for each gluon that doesn't flip the spin,
  - v^3 for each gluon that flips the spin.
- The Fock-state expansion is sometimes expressed loosely as a statement about amplitudes:

$$\begin{aligned} |\text{quarkonium}\rangle &= A_{Q\bar{Q}}|Q\bar{Q}\rangle + A_{Q\bar{Q}g}|Q\bar{Q}g\rangle \\ &\quad + A_{Q\bar{Q}gg}|Q\bar{Q}gg\rangle + A_{Q\bar{Q}q\bar{q}}|Q\bar{Q}q\bar{q}\rangle + \ldots . \end{aligned}$$

But it is really a statement about probabilities (the squares of amplitudes integrated over phase space).

Space-Time Picture of Heavy-Quarkonium Annihilation



- The points $A(C)$ and $B(D)$ are within $\sim 1/m$ of each other.
  - Emission of a hard gluon puts the Q or $\bar{Q}$ into a highly virtual state.
  - Virtuality $\sim m$ implies propagation over distance $\sim 1/m$.
- A and C are within $1/m$ of each other.
  - Somewhat surprising since the outgoing gluons (jets) are on shell.

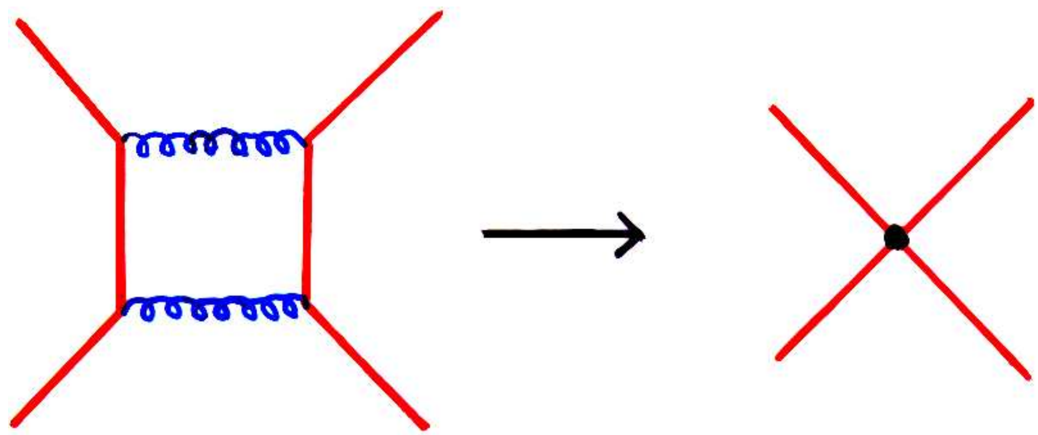
- But the energies/momenta of the jets are large and the gluons are exactly on shell.
  * The squared amplitude is insensitive to changes of external momentum of order m.
  * The Fourier transform has support over a distance of order $1/m$.
- Classically, can trace the final-state jets back to the annihilation vertex.
- In quantum mechanics, there is an uncertainty of $\sim 1/p \sim 1/m$ in the location of the vertex.

• Soft final-state interactions could spoil this argument.

- The gluon could propagate a long distance in an arbitrary direction before the jet emerges.
- Soft divergences cancel by the KLN thm. for inclusive processes.

$Q\bar{Q}$ Annihilation in NRQCD

- The size of the annihilation vertex is $\sim 1/m \sim 1/\Lambda$.
- In NRQCD the annihilation is represented by local 4-fermion interactions.



- The finite size of the annihilation vertex is taken into account by including operators of higher order in v.
- Because of annihilation, probability is not conserved in NRQCD.
  - The coefficient of the 4-fermion interaction f_n has an imaginary part.
- The annihilation rate is given by the NRQCD factorization formula, which follows from the optical theorem:

$$\Gamma(H \to \mathrm{LH}) = \sum_n \frac{2 \, \mathrm{Im} \, f_n(\Lambda)}{m^{d_n - 4}} \langle H | \mathcal{O}_n(\Lambda) | H \rangle .$$

- Determine the f_n by matching annihilation amplitudes (on-shell) between full QCD and NRQCD.
  - The f_n are short-distance coefficients.
  - The matching can be done perturbatively.
- All of the nonperturbative physics is in matrix elements of the 4-fermion operators in the quarkonium state.
  - Analogous to parton distributions.
  - Calculate on the lattice or determine from experiments.
  - Heavy-quark spin symmetry and vacuum-saturation approximation allow application to more than one decay process.

- Annihilation operators of dimension 6:

$$
\begin{aligned}
\mathcal{O}_1({}^1S_0) &= \psi^\dagger \chi \, \chi^\dagger \psi, \\
\mathcal{O}_1({}^3S_1) &= \psi^\dagger \boldsymbol{\sigma} \chi \cdot \chi^\dagger \boldsymbol{\sigma} \psi, \\
\mathcal{O}_8({}^1S_0) &= \psi^\dagger T^a \chi \, \chi^\dagger T^a \psi, \\
\mathcal{O}_8({}^3S_1) &= \psi^\dagger \boldsymbol{\sigma} T^a \chi \cdot \chi^\dagger \boldsymbol{\sigma} T^a \psi.
\end{aligned}
$$

- The $Q\bar{Q}$ pair can annihilate in a color-octet or a color-singlet.
- The color-octet matrix element is proportional to the probability to find a $Q\bar{Q}g$ Fock state.
  - Suppressed by powers of v, but can be important for P-wave quarkonium decay.
  - The decay operators that connect to the leading Fock state may also be suppressed by powers of v.
- If we drop all of the color-octet contributions and retain only the color-singlet contribution that is leading in v, then we have the color-singlet model.
- In contrast, NRQCD factorization for decays is a rigorous consequence of QCD in the limit $m \gg \Lambda_{\mathrm{QCD}}$.
- Because of uncanceled IR divergences, the color-singlet model is inconsistent in the treatment of P-wave states.

Application of the Heavy-Quark Spin Symmetry

- The heavy-quark spin symmetry gives relations between matrix elements for quarkonium states that are related by a spin flip.
- Examples for color-singlet matrix elements:

$$\begin{aligned} \langle \psi | \mathcal{O}_1(^3S_1) | \psi \rangle &= \langle \eta_c | \mathcal{O}_1(^1S_0) | \eta_c \rangle, \\ \langle \chi_{cJ} | \mathcal{O}_1(^3P_J) | \chi_{cJ} \rangle &= \langle h_c | \mathcal{O}_1(^1P_1) | h_c \rangle, \quad J = 0, 1, 2. \end{aligned}$$

- Examples for color-octet matrix elements:

$$\langle \chi_{cJ} | \mathcal{O}_8(^3S_1) | \chi_{cJ} \rangle = \langle h_c | \mathcal{O}_8(^1S_0) | h_c \rangle, \quad J = 0, 1, 2.$$

- These relations hold up to corrections of order v^2.

Vacuum-Saturation Approximation

- For a color-singlet operator $\mathcal{O}_n$, insert a complete set of intermediate states in an operator matrix element:

$$\langle H|\mathcal{O}_n|H\rangle = \sum_X \langle H|\psi^\dagger \kappa_n' \chi|X\rangle\langle X|\chi^\dagger \kappa_n \psi|H\rangle.$$

- Retain only the vacuum intermediate state:

$$\langle H|\mathcal{O}_n|H\rangle \approx \langle H|\psi^\dagger \kappa_n' \chi|0\rangle\langle 0|\chi^\dagger \kappa_n \psi|H\rangle.$$

- The leading color-singlet intermediate state contains two gluons.
  - Suppressed as $(v^2)^2 = v^4$.
  - Therefore, the vacuum-saturation approximation for color-singlet operators holds up to corrections of order v^4.
- The matrix elements that appears in EM decays are identical to the vacuum-saturation expressions.
  - Gives additional predictive power.

Relation to the Bethe-Salpeter Wave Function

- Example: the Bethe-Salpeter wave function (Coulomb gauge) for the η_c is

$$\Psi_{\eta_c}(\boldsymbol{x}) = \frac{1}{\sqrt{2N_c}} \langle 0 | \chi^\dagger(-\boldsymbol{x}/2)\psi(\boldsymbol{x}/2) | \eta_c \rangle .$$

  - The factor $1/\sqrt{2N_c}$ takes into account the sum over the spin and color degrees of freedom in the normalization condition.

- Therefore, the wave function at the origin is related to the vacuum-saturation approximation for the NRQCD matrix element:

$$\Psi_{\eta_c}(0) = \frac{1}{\sqrt{2N_c}} \langle 0 | \chi^\dagger \psi | \eta_c \rangle .$$

- Similarly, for the J/ψ,

$$\Psi_{J/\psi}(0)\,\boldsymbol{\epsilon} = \frac{1}{\sqrt{2N_c}} \langle 0 | \chi^\dagger \boldsymbol{\sigma} \psi | J/\psi(\boldsymbol{\epsilon}) \rangle ,$$

  where $\boldsymbol{\epsilon}$ is the polarization vector of the J/ψ.

Examples of Matching in Annihilation Decay

η_c Decay

- At leading order in v, η_c annihilation decay proceeds through a color-singlet matrix element:

$$\Gamma(\eta_c \to \mathrm{LH}) = \frac{2\ \mathrm{Im}\ f_1(^1S_0)}{m^2}\ \langle \eta_c | \mathcal{O}_1(^1S_0) | \eta_c \rangle .$$
$$\mathcal{O}_1(^1S_0) = \psi^\dagger \chi\, \chi^\dagger \psi .$$

- That is, we have the color-singlet-model result.
- Determine $\mathrm{Im}\, f_1(^1S_0)$ by matching NRQCD and full QCD matrix elements in $Q\bar{Q}$ states.
- At leading order in α_s, a $Q\bar{Q}$ pair in a $^1S_0$ color-singlet state decays into two gluons. The matching condition is

$$\frac{2\,\mathrm{Im}\, f_1^{(0)}(^1S_0)}{m^2} \langle Q\bar{Q}_1(^1S_0) | \mathcal{O}_1(^1S_0) | Q\bar{Q}_1(^1S_0) \rangle^{(0)}$$
$$= \hat{\Gamma}^{(0)}[Q\bar{Q}_1(^1S_0) \to gg)].$$

.

  - With a suitable normalization of the states,

$$\langle Q\bar{Q}_1(^1S_0) | \mathcal{O}_1(^1S_0) | Q\bar{Q}_1(^1S_0) \rangle^{(0)} = 1 .$$

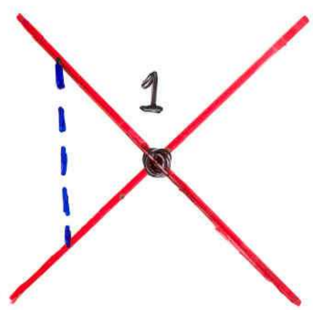
  - Therefore,

$$\frac{2\,\mathrm{Im}\, f_1^{(0)}(^1S_0)}{m^2} = \hat{\Gamma}^{(0)}[Q\bar{Q}(^1S_0) \to gg].$$

- At next-to-leading order in α_s, the matching condition for a $Q\bar{Q}$ pair in a color-singlet ${}^1S_0$ state is

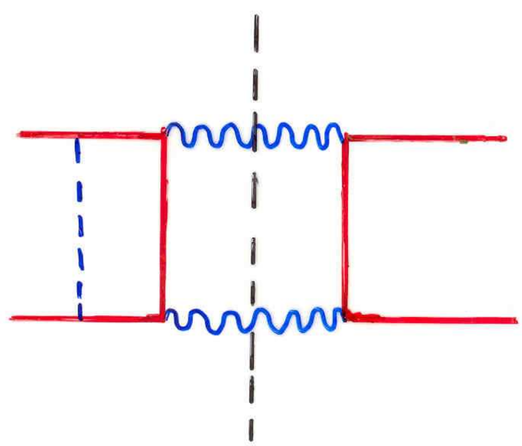
$$
\begin{aligned}
&\frac{2\ \mathrm{Im}\, f_1^{(1)}({}^1S_0)}{m^4}\ \langle Q\bar{Q}_1({}^1S_0)|\mathcal{O}_1({}^1S_0)|Q\bar{Q}_1({}^1S_0)\rangle^{(0)} \\
&+\frac{2\ \mathrm{Im}\, f_1^{(0)}({}^1S_0)}{m^4}\ \langle Q\bar{Q}_1({}^1S_0)|\mathcal{O}_1({}^1S_0)|Q\bar{Q}_1({}^1S_0)\rangle^{(1)} \\
&= \hat{\Gamma}^{(0)}[Q\bar{Q}_1({}^1S_0) \to ggg] + \hat{\Gamma}^{(1)}[Q\bar{Q}_1({}^1S_0) \to gg] \\
&\quad +\hat{\Gamma}^{(0)}[Q\bar{Q}_1({}^1S_0) \to q\bar{q}g].
\end{aligned}
$$

- We wish to solve for $\mathrm{Im}\, f_1^{(1)}({}^1S_0)$.
- $\langle Q\bar{Q}_1({}^1S_0)|\mathcal{O}_1({}^1S_0)|Q\bar{Q}_1({}^1S_0)\rangle^{(1)}$ contains one-loop corrections that don't change the color of the initial or final $Q\bar{Q}$ pair:

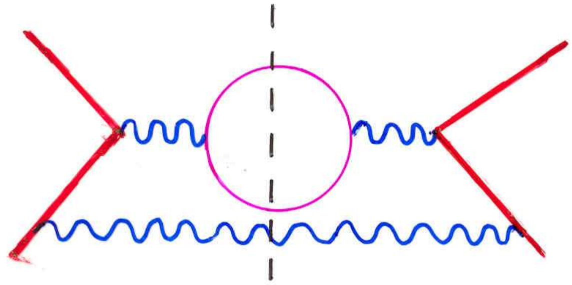


The gluon connects to only the initial Q or $\bar{Q}$ or the final Q or $\bar{Q}$.

- $\hat{\Gamma}^{(0)}[Q\bar{Q}_1({}^1S_0) \to ggg]$ contains real corrections to $Q\bar{Q}_1({}^1S_0) \to gg$.
  - Its logarithmic IR and collinear divergences are canceled by similar divergences from virtual corrections in $\hat{\Gamma}^{(1)}[Q\bar{Q}_1({}^1S_0) \to gg]$.
- $\hat{\Gamma}^{(1)}[Q\bar{Q}_1({}^1S_0) \to gg]$ contains virtual corrections to $Q\bar{Q}_1({}^1S_0) \to gg$.
  - Its logarithmic IR and collinear divergences are canceled by real corrections in $\hat{\Gamma}(0)[Q\bar{Q}_1({}^1S_0) \to ggg]$ and $\hat{\Gamma}^{(0)}[Q\bar{Q}_1({}^1S_0) \to q\bar{q}g]$.
  - It also contains a power infrared divergence that is associated with exchange of a Coulomb gluon between the initial or final $Q\bar{Q}$ pair.

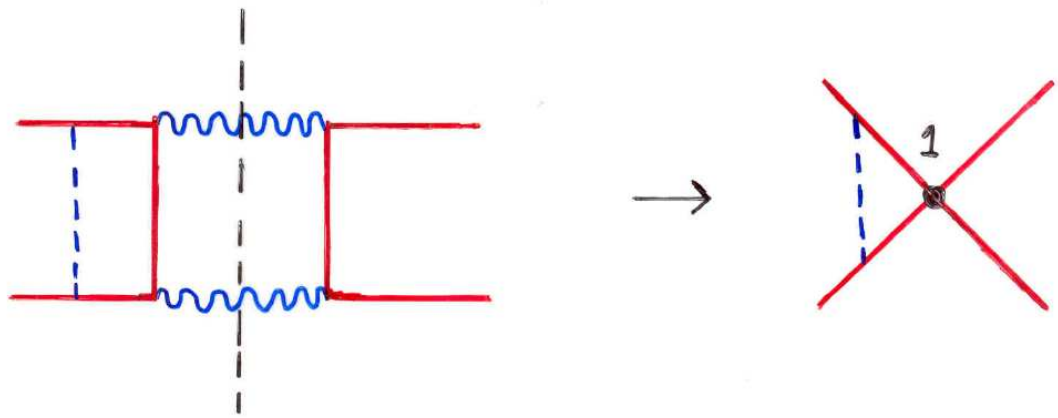


- $\hat{\Gamma}^{(0)}[Q\bar{Q}_1({}^1S_0) \to q\bar{q}g]$ is the Born process for $Q\bar{Q}_1({}^1S_0) \to q\bar{q}g$.



  - The gluon is emitted from the Q or $\bar{Q}$ (C parity).
  - It contains collinear divergences that are canceled by collinear divergences in the virtual corrections in $\hat{\Gamma}^{(1)}[Q\bar{Q}_1({}^1S_0) \to gg]$.

- NRQCD reproduces full QCD at small momenta.
  - The loop correction to the color-singlet matrix element cancels the power IR divergence in $\hat{\Gamma}^{(1)}[Q\bar{Q}_1(^1S_0) \to gg]$.



Interpretation: The Coulomb-gluon correction is absorbed into a re-definition of the $Q\bar{Q}$ wave function.

- Therefore, $\mathrm{Im}\, f_1^{(1)}(^1S_0)$ is IR finite.

χ_{c0} Decay

- At leading order in v, χ_{c0} decay proceeds through both color-singlet and color-octet $Q\bar{Q}$ states:

$$\begin{aligned}\Gamma(\chi_{c0} \to \mathrm{LH}) &= \frac{2\ \mathrm{Im}\, f_1(^3P_0)}{m^4} \langle \chi_{c0} | \mathcal{O}_1(^3P_0) | \chi_{c0} \rangle \\ &\quad + \frac{2\ \mathrm{Im}\, f_8(^3S_1)}{m^2} \langle \chi_{c0} | \mathcal{O}_8(^3S_1) | \chi_{c0} \rangle .\end{aligned}$$

$$\begin{aligned}\mathcal{O}_1(^3P_0) &= \frac{1}{3}\, \psi^\dagger(-\tfrac{i}{2}\overleftrightarrow{\mathbf{D}} \cdot \boldsymbol{\sigma})\chi\, \chi^\dagger(-\tfrac{i}{2}\overleftrightarrow{\mathbf{D}} \cdot \boldsymbol{\sigma})\psi \\ \mathcal{O}_8(^3S_1) &= \psi^\dagger \boldsymbol{\sigma} T^a \chi \cdot \chi^\dagger \boldsymbol{\sigma} T^a \psi .\end{aligned}$$

- $\mathcal{O}_1(^3P_0)$ connects to the leading χ_{c0} Fock state, but has two derivatives, which become two powers of v.
- $\mathcal{O}_8(^3S_1)$ has no derivatives, but it connects to the $Q\bar{Q}g$ Fock state, which is suppressed by two powers of v in the probability.
- Therefore, both matrix elements are of the same order in v.

- At leading order in α_s, a $Q\bar{Q}$ pair in a color-singlet ${}^3P_0$ state decays into two gluons. The matching condition is

$$\frac{2\ \mathrm{Im}\ f_1^{(0)}({}^3P_0)}{m^4}\ \langle Q\bar{Q}_1({}^3P_0)|\mathcal{O}_1({}^3P_0)|Q\bar{Q}_1({}^3P_0)\rangle^{(0)}$$
$$= \hat{\Gamma}^{(0)}[Q\bar{Q}_1({}^3P_0) \to gg].$$

- At leading order in α_s, a $Q\bar{Q}$ pair in a color-octet ${}^3S_1$ state decays through a virtual gluon into a light $q\bar{q}$ pair. The matching condition is

$$\frac{2\ \mathrm{Im}\ f_8^{(0)}({}^3S_1)}{m^4}\ \langle Q\bar{Q}_8({}^3S_1)|\mathcal{O}_8({}^3S_1)|Q\bar{Q}_8({}^3S_1)\rangle^{(0)}$$
$$= \hat{\Gamma}^{(0)}[Q\bar{Q}_8({}^3S_1) \to q\bar{q}].$$

- Again, the $Q\bar{Q}$ states can be normalized so that the matrix elements are unity.

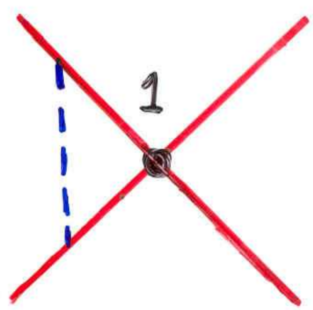
- Then, we have for the short-distance coefficients

$$\frac{2\ \mathrm{Im}\ f_1^{(0)}({}^3P_0)}{m^4} = \hat{\Gamma}^{(0)}[Q\bar{Q}_1({}^3P_0) \to gg],$$
$$\frac{2\ \mathrm{Im}\ f_8^{(0)}({}^3S_1)}{m^4} = \hat{\Gamma}^{(0)}[Q\bar{Q}_8({}^3S_1) \to q\bar{q}].$$

- At next-to-leading order in α_s, the matching condition for a $Q\bar{Q}$ pair in a color-singlet ${}^3P_0$ state is

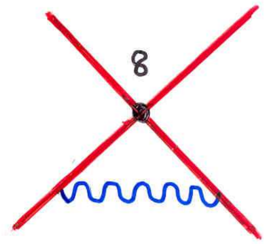
$$
\begin{aligned}
&\frac{2\ \mathrm{Im}\ f_1^{(1)}({}^3P_0)}{m^4}\ \langle Q\bar{Q}_1({}^3P_0)|\mathcal{O}_1({}^3P_0)|Q\bar{Q}_1({}^3P_0)\rangle^{(0)} \\
&+\frac{2\ \mathrm{Im}\ f_1^{(0)}({}^3P_0)}{m^4}\ \langle Q\bar{Q}_1({}^3P_0)|\mathcal{O}_1({}^3P_0)|Q\bar{Q}_1({}^3P_0)\rangle^{(1)} \\
&+\frac{2\ \mathrm{Im}\ f_8^{(0)}({}^3S_1)}{m^4}\ \langle Q\bar{Q}_1({}^3P_0)|\mathcal{O}_8({}^3S_1)|Q\bar{Q}_1({}^3P_0)\rangle^{(1)} \\
&= \hat{\Gamma}^{(0)}[Q\bar{Q}_1({}^3P_0) \to ggg] + \hat{\Gamma}^{(1)}[Q\bar{Q}_1({}^3P_0) \to gg] \\
&\quad +\hat{\Gamma}^{(0)}[Q\bar{Q}_1({}^3P_0) \to q\bar{q}g].
\end{aligned}
$$

- We wish to solve for $\mathrm{Im}\ f_1^{(1)}({}^3P_0)$.
- $\langle Q\bar{Q}_1({}^3P_0)|\mathcal{O}_1({}^3P_0)|Q\bar{Q}_1({}^3P_0)\rangle^{(1)}$ contains one-loop corrections that don't change the color of the initial or final $Q\bar{Q}$ pair:



The gluon connects to only the initial Q or $\bar{Q}$ or the final Q or $\bar{Q}$.

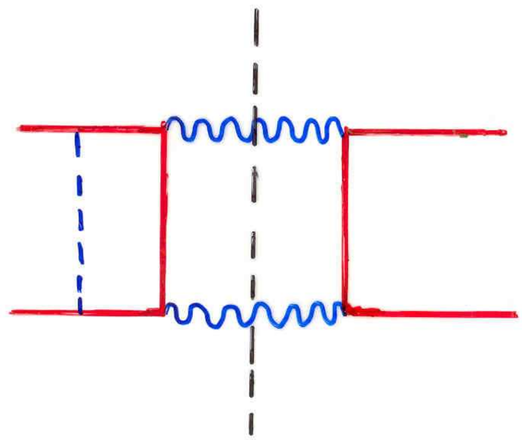
- $\langle Q\bar{Q}_1(^3P_0)|\mathcal{O}_8(^3S_1)|Q\bar{Q}_1(^3P_0)\rangle^{(1)}$ contains one-loop corrections that put the initial and final $Q\bar{Q}$ pairs in $^3S_1$ color-octet states:



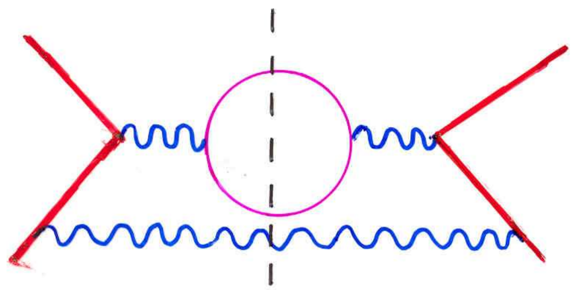
The gluon connects the initial Q or $\bar{Q}$ to the final Q or $\bar{Q}$. It carries off one unit of orbital angular momentum.

- $\hat{\Gamma}^{(0)}[Q\bar{Q}_1(^3P_0) \to ggg]$ contains real corrections to $Q\bar{Q}_1(^3P_0) \to gg$.
  - Its logarithmic IR and collinear divergences are canceled by similar divergences from virtual corrections in $\hat{\Gamma}^{(1)}[Q\bar{Q}_1(^3P_0) \to gg]$.

- $\hat{\Gamma}^{(1)}[Q\bar{Q}_1(^3P_0) \to gg]$ contains virtual corrections to $Q\bar{Q}_1(^3P_0) \to gg$.
  - Its logarithmic IR and collinear divergences are canceled by real corrections in $\hat{\Gamma}(0)[Q\bar{Q}_1(^3P_0) \to ggg]$ and $\hat{\Gamma}^{(0)}[Q\bar{Q}_1(^3P_0) \to q\bar{q}g]$.
  - It also contains a power infrared divergence that is associated with exchange of a Coulomb gluon between the initial or final $Q\bar{Q}$ pair.

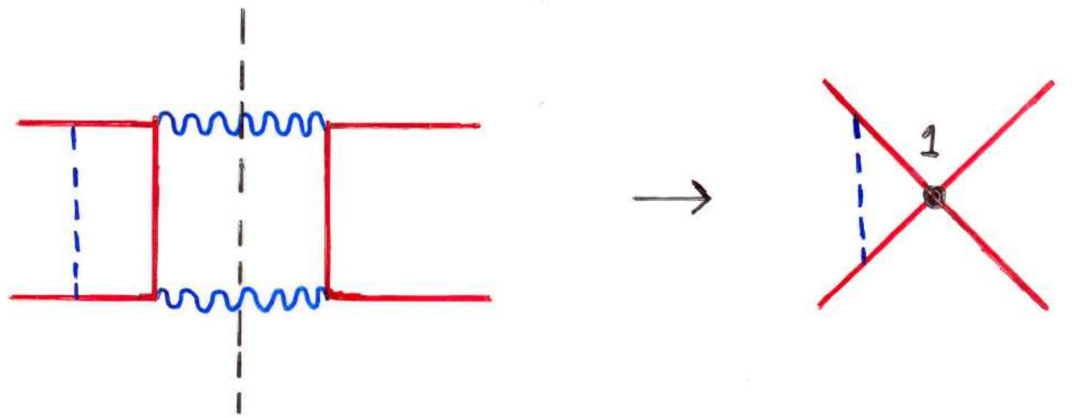


- $\hat{\Gamma}^{(0)}[Q\bar{Q}_1(^3P_0) \to q\bar{q}g]$ is the Born process for $Q\bar{Q}_1(^3P_0) \to q\bar{q}g$.



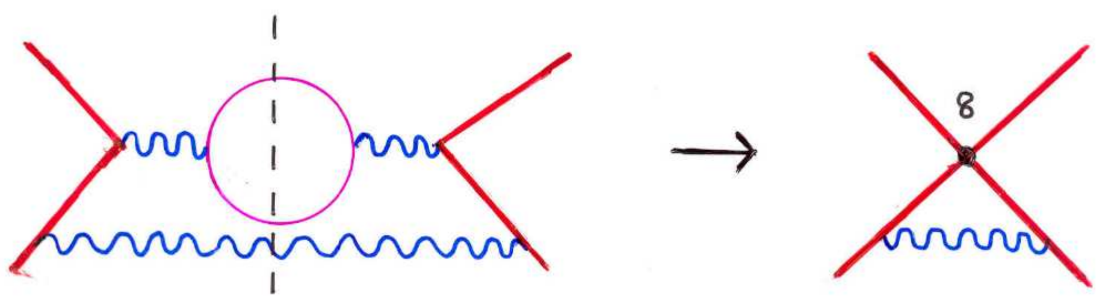
  - The gluon is emitted from the Q or $\bar{Q}$ (C parity).
  - It contains collinear divergences that are canceled by collinear divergences in the virtual corrections in $\hat{\Gamma}^{(1)}[Q\bar{Q}_1(^3P_0) \to gg]$.
  - After all cancellations among full-QCD processes, it still contains a logarithmic infrared divergence.
    * This was a long-standing puzzle in the color-singlet model.

- NRQCD reproduces full QCD at small momenta.
  - The loop correction to the color-singlet matrix element cancels the power IR divergence in $\hat{\Gamma}^{(1)}[Q\bar{Q}_1(^3P_0) \to gg]$.



Interpretation: The Coulomb-gluon correction is absorbed into a re-definition of the $Q\bar{Q}$ wave function.

  - The loop correction to the color-octet matrix element cancels the logarithmic IR divergence in $\hat{\Gamma}^{(0)}[Q\bar{Q}_1(^3P_0) \to q\bar{q}g]$.



Interpretation: The real gluon is part of the QQg Fock state when its momentum is less than Λ. Otherwise, it contributes to the short-distance coefficient.

- Therefore, $\mathrm{Im}\, f_1^{(1)}({}^3P_0)$ is IR finite.
- In the color-singlet model, the color-octet matrix element is absent, and the short-distance coefficient contains a logarithmic IR divergence.
- This inconsistency in the color-singlet model was the original motivation for the development of the NRQCD factorization formalism.
- There is a similar matching condition for a $Q\bar{Q}$ pair in a color-octet ${}^3S_1$ state that can be used to determine $\mathrm{Im}\, f_8^{(1)}({}^3S_1)$.

- We would like to apply NRQCD methods to heavy-quarkonium production processes.
- The probability for a $Q\bar{Q}$ pair to evolve into a heavy quarkonium can be calculated as a vacuum-matrix element in NRQCD:

$$\begin{aligned}&\langle 0|\mathcal{O}_n^H|0\rangle\\&= \langle 0|\chi^\dagger \kappa_n \psi \left(\sum_X |H+X\rangle\langle H+X|\right)\psi^\dagger \kappa_n' \chi|0\rangle.\end{aligned}$$

- This is the matrix element of a four-fermion operator, but with a projection onto an intermediate state of the quarkonium H plus anything.

- The production matrix elements are the crossed versions of quarkonium decay matrix elements.
  - Only the color-singlet production and decay matrix elements are simply related by the vacuum-saturation approximation.
  - Replace X with the vacuum in the color-singlet production matrix elements:

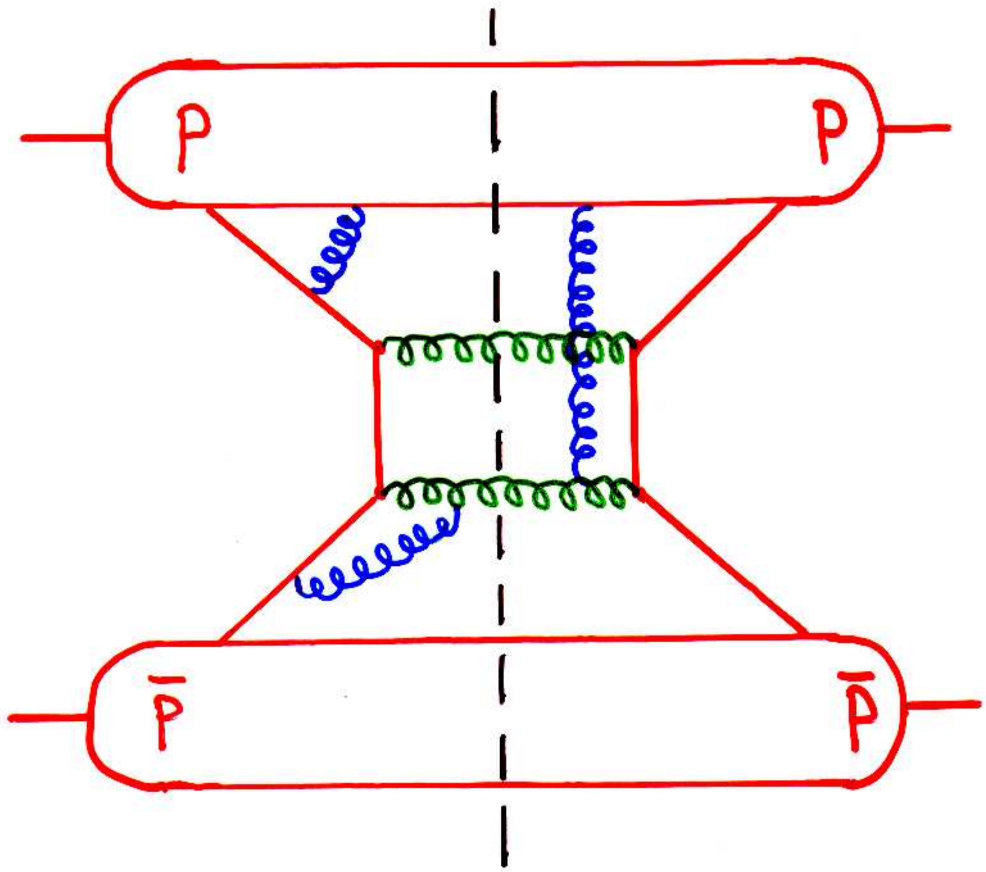
$$\langle 0|\mathcal{O}_n^H|0\rangle \approx \langle 0|\chi^\dagger \kappa_n \psi|H\rangle\langle H|\psi^\dagger \kappa_n' \chi|0\rangle .$$

    These are same amplitudes that appear in the decay matrix elements in the vacuum-saturation approximation.

- In order to prove that the operator is a *local* product of four heavy-quark operators, we have to establish that the $Q\bar{Q}$ production process occurs at short distances (of order $1/m$ or $1/p_T$).

- To establish the short-distance nature of the production process and to prove factorization, we must show that
  - soft and collinear divergences cancel or can be absorbed into parton distributions, fragmentation functions, or NRQCD matrix elements (short-distance process);
  - spectator interactions cancel or can be absorbed into parton distributions fragmentation functions, or NRQCD matrix elements (topological factorization).
- There is an existing technology for demonstrating this to all orders in perturbation theory.

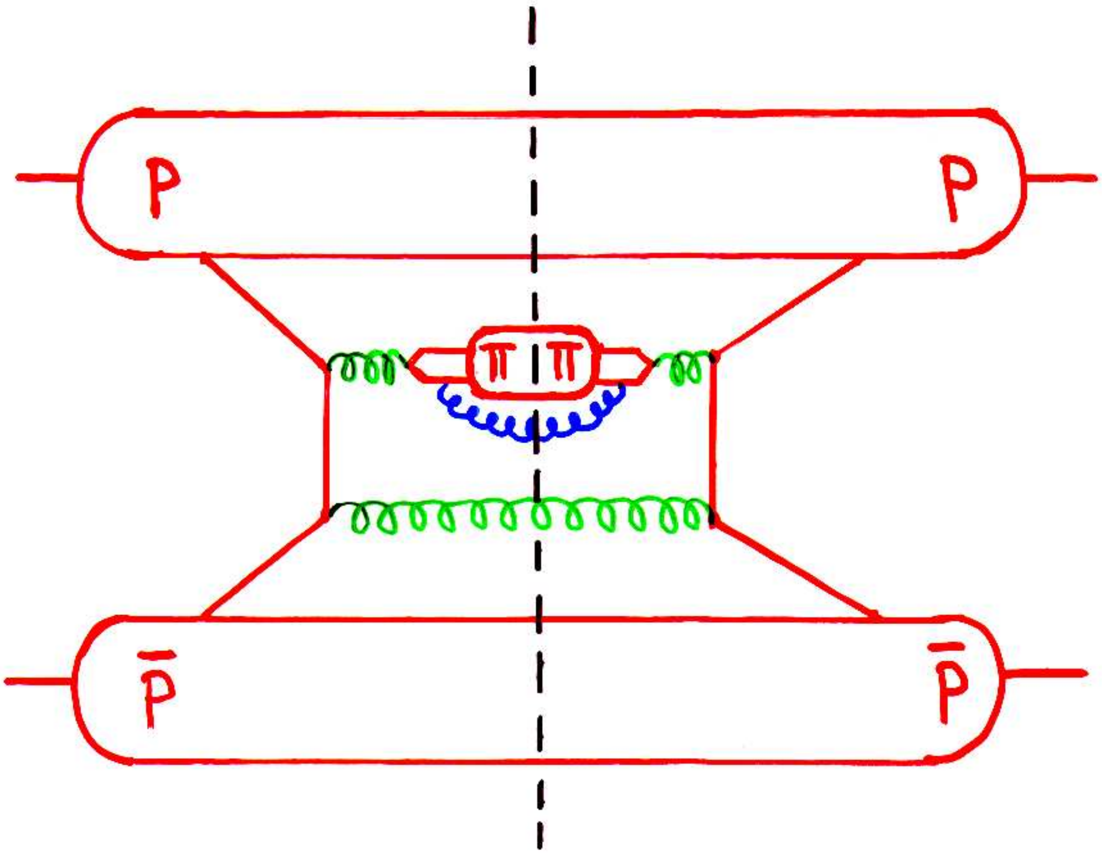
Jet Production at Large p_T



- The simple parton model is complicated in QCD by additional gluon exchanges, possibly involving spectator quarks.
- Many years ago, the technology was developed for dealing with the gluon exchanges to all orders in perturbation theory for sufficiently inclusive processes (Collins, Soper, Sterman; GTB; many others).

- Result at leading order in $1/p_T$:
  - Soft gluons factor completely from the original diagram.
  - Collinear gluons factor into separate contributions for each beam or jet direction.
  - Initial-state collinear contributions factor into PDF's.
  - Factored soft and final-state collinear contributions cancel by unitarity.
  - The remaining hard contributions involve only active partons and can be calculated in QCD perturbation theory.
  - The cross section is a convolution of PDF's with perturbatively calculable partonic cross sections.
  - Corrections to this result:
    - Order $\Lambda^2_{\mathrm{QCD}}/p_T^2$ in the unpolarized case,
    - Order $\Lambda_{\mathrm{QCD}}/p_T$ in the polarized case.

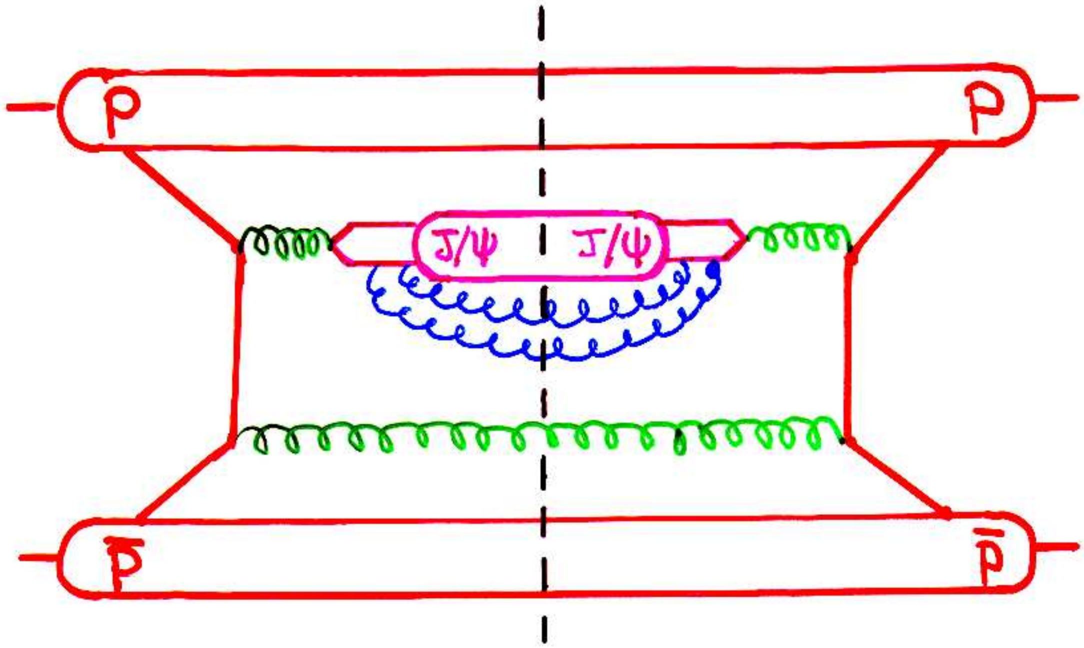
Inclusive Single-Particle Production at Large p_T



- Radiated soft gluons and/or light quarks turn the final-state parton into a light hadron.
- The cross section is no longer completely inclusive, so the unitarity cancellation of contributions from gluons collinear to the final hadron fails.

- These collinear contributions are factored into the fragmentation function for the parton to become a hadron.
- The cross section is a convolution of PDF's and the fragmentation function with a perturbatively calculable partonic cross section.
- Corrections to this result:
  - Order $\Lambda_{\mathrm{QCD}}^2/p_T^2$ in the unpolarized case,
  - Order $\Lambda_{\mathrm{QCD}}/p_T$ in the polarized case.

Inclusive Heavy-Quarkonium Production at Large p_T



- A detailed proof does not exist for this case. (Qiu, Sterman, work in progress)
- It seems likely that the standard technology can be applied to prove factorization.
- Again, the cross section is no longer completely inclusive, so the unitarity cancellation of contributions from gluons collinear to the quarkonium fails.
- These collinear contributions are soft contributions in the quarkonium CM frame.
- They are factored into the NRQCD matrix element.

- Best guess for corrections to this result:
  - order $\Lambda_{\rm QCD}^2/p_T^2$ (not m_Q^2/p_T^2) in the unpolarized case,
  - order $\Lambda_{\rm QCD}/p_T$ (not m_Q/p_T) in the polarized case.
- It is not known if there is a factorization formula at small p_T.
  - If there is factorization, it would likely hold only in leading order in v (Qiu, Sterman):
    The Bloch-Nordsieck IR cancellation fails in order v (Doria, Frenkel, Taylor; Di'Lieto, Gendron, Halliday, Sachrajda).
  - Since m is now the large scale, there are probably violations of factorization of order $\Lambda_{\rm QCD}^2/m^2$ (unpolarized) and $\Lambda_{\rm QCD}/m$ (polarized).
- The cross section can be written as a sum of products of NRQCD matrix elements and "short-distance" coefficients:

$$\sigma(H) = \sum_n \frac{F_n(\Lambda)}{m^{d_n-4}} \langle 0|\mathcal{O}_n^H(\Lambda)|0\rangle .$$

- The "short-distance" coefficients $F_n(\Lambda)$ are essentially the process-dependent partonic cross sections to make a $Q\bar{Q}$ pair convolved with the parton distributions.
  - They have an expansion in powers of α_s.

- The sum over matrix elements is an expansion in powers of v.
  In practice, truncate it at a finite order.

- The operator matrix elements are universal (process independent).
  Universality gives NRQCD factorization much of its predictive power.

- Choose the NRQCD cutoff Λ (NRQCD factorization scale) large enough so that $mv < \Lambda \lesssim m$, but $\alpha_s(\Lambda) << 1$.
  - Gluons with $k < m$ are included in the matrix element.
  - Gluons with $k > m$ are part of the short-distance coefficients and are treated perturbatively.

- NRQCD factorization relies on NRQCD and hard-scattering factorization.
  Comparisons with experiment test both.

- An important feature of NRQCD factorization:
  Quarkonium production occurs through color-octet, as well as color-singlet, $Q\bar{Q}$ states.

- If we drop all of the color-octet contributions, then we have the color-singlet model (CSM).

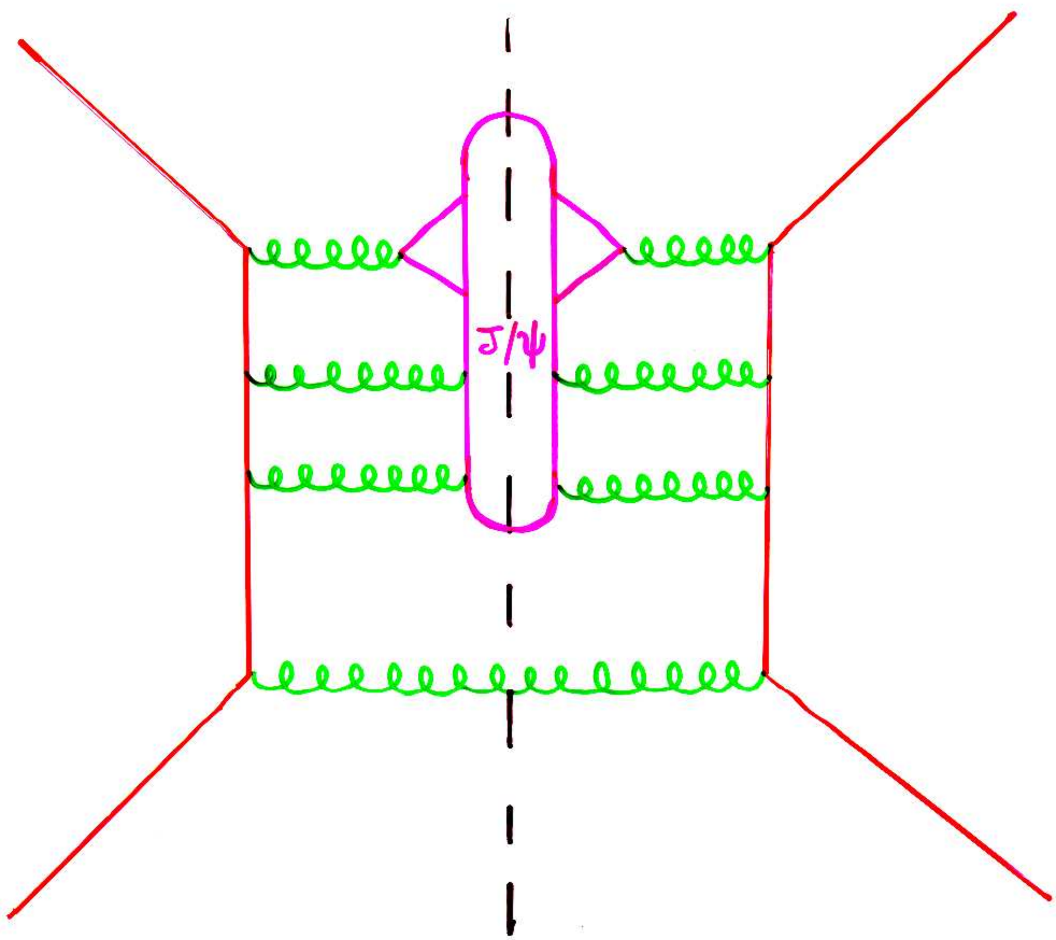
- In contrast, NRQCD factorization for production is not a model.
  - Sometimes erroneously called "the color-octet model."
  - Believed to be a rigorous consequence of QCD in the limit m, $p_T \gg \Lambda_{\mathrm{QCD}}$.

- Because of uncanceled IR divergences, the color-singlet model is inconsistent in the treatment of production of P-wave states.

Gluon Radiation in Production Matrix Elements

- As in decays
  - Emission of a gluon that doesn't flip the spin costs v^2 in probability;
  - Emission of a gluon that flips the spin costs v^3 in probability.
- In quarkonium production, emission of gluons from produced $Q\bar{Q}$ pairs can be important
  - if the production operators with the quantum numbers of the quarkonium are suppressed by powers of v,
  - if production of a $Q\bar{Q}$ state with the quantum numbers of the quarkonium is kinematically suppressed.

- A common misconception: color-octet production proceeds through a higher Fock state.
  - In leading color-octet production, the gluons that neutralize the color are in the final state, not the initial state.
- The higher Fock-state process requires the production of gluons that are nearly collinear to the $Q\bar{Q}$ pair:



- It is suppressed by additional powers of v.

Summary

- Quarkonia are multi-scale systems.
- The effective field theory NRQCD can be used to separate
  - short-distance perturbative scales with momenta of order m and higher,
  - long-distance non-perturbative scales with momenta less than m.
- NRQCD is constructed by integrating out the high-momentum modes in QCD.
- The NRQCD action is an expansion in powers of the heavy-quark–antiquark relative velocity v.
- Inclusive quarkonium decay and production rates are given in NRQCD as a sum of matrix elements of local four-fermion operators times short-distance coefficients.
  - The coefficients can be calculated in perturbation theory by matching amplitudes in full QCD with those in NRQCD.
  - The sum over matrix elements is an expansion in powers of v that, in practice, is truncated.
- The NRQCD factorization formula for production relies on both NRQCD and hard-scattering factorization.

- Production and decay proceed through color-octet, as well as color-singlet $Q\bar{Q}$ channels.
- The heavy-quark spin symmetry and the vacuum-saturation approximation can be used to obtain approximate relations between matrix elements of NRQCD four-fermion operators.
- Only color-singlet production and decay matrix elements are simply related.
- Matrix elements of production operators are universal (process independent).
- The NRQCD factorization formulas for decay and production are not models, but are consequences of QCD in the limits $m, p_T \gg \Lambda_{\mathrm{QCD}}$.